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SMALE'S HORSESHOE MAP VIA TERNARY NUMBERS*

JOHN BANKS[†] AND VALENTINA DRAGAN[†]

Abstract. Smale's horseshoe map has become a standard example in the study of discrete dynamical systems. Smale's horseshoe map occurs in a wide range of physical problems with chaotic dynamics on an invariant set. Its chaotic behavior is usually shown by first conjugating it to the two-sided shift map. The authors give a more elementary treatment of a special, but typical case. The only technical background needed is the Cantor set and its ternary representation.

Key words. Smale's Horseshoe, chaos, dynamical systems, Cantor set, dense orbit, dense periodic points, sensitivity to initial conditions

AMS subject classifications. 58F13, 34C35, 26A18

1. Introduction. The analysis of the van der Pol oscillator given by Cartwright and Littlewood, and by Levinson led Smale to invent the horseshoe map [6], [7]. Smale's horseshoe exhibited the chaotic behavior of the van der Pol oscillator but, because of its greater simplicity, he was able to prove structural stability. The horseshoe is now known to be present in many systems (see, for example, [3], [4], and [8]).

Here we understand chaos in accordance with Devaney's definition: the map should have a *dense orbit*, *dense periodic points*, and *sensitivity to initial conditions* [2]. The usual method involves a rather difficult inductive argument using the geometry of the higher iterates of the horseshoe map (see Wiggins [8, p. 423] for a complete account).

In this paper the two-dimensional problem for the horseshoe map is solved using an approach that is an easy extension of that used for the one-dimensional tent map and is similar to that given by Peitgen et al. [5, p. 80] for the binary representation of a related tent map. Symbolic dynamics arises naturally from our use of ternary representation of numbers. This approach allows us to introduce the horseshoe map to students somewhat earlier than usual.

A tent map on the Cantor set. To motivate our treatment of the horseshoe map, we first study a simpler tent map. We define $T : \mathbb{R} \rightarrow \mathbb{R}$ by putting

$$T(x) = \begin{cases} 3x & \text{if } x \leq \frac{1}{2}, \\ 3 - 3x & \text{if } x > \frac{1}{2}. \end{cases}$$

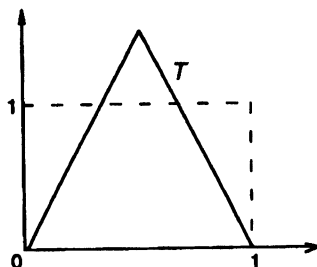


FIG. 1. The tent map acting on the unit interval.

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We study the dynamics of the map T on the set Σ of all points x whose iterates stay in $[0,1]$ (since for any other point the iterates approach $-\infty$).

Figure 2 suggests how to obtain Σ . First remove the middle third of $[0, 1]$ and then remove the middle third from the remaining intervals, and so on. This is the classic construction for the *Cantor Middle-Thirds Set*.

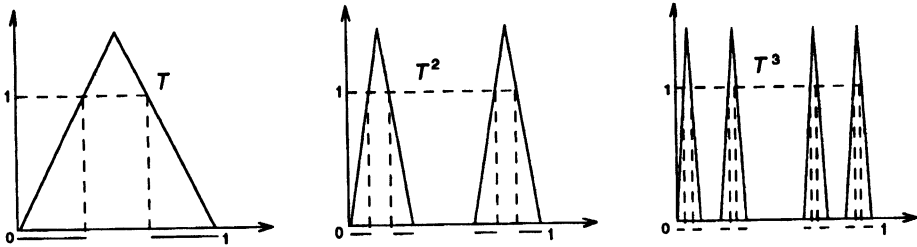


FIG. 2. Intervals mapping into $[0, 1]$ under T, T^2, T^3 .

To actually prove that Σ is the Cantor set we use the representation of the Cantor set in ternary numbers. We adopt the convention that, where there exist two alternative representations of a given number, we choose the form that ends with an infinite string of 2's, rather than the form that ends with a 1 followed by an infinite string of 0's. The Cantor set then consists of the numbers in $[0,1]$ whose ternary expansions contain only 0's and 2's.

Let $x = 0.x_1x_2x_3 \dots$ in ternary and introduce the notation $\bar{x}_i = 2 - x_i$. Multiplication by 3 shifts x one digit to the left, and $3 = 2.222 \dots$ in ternary. Hence

$$T(x) = \begin{cases} x_1.x_2x_3x_4 \dots & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \bar{x}_1.\bar{x}_2\bar{x}_3\bar{x}_4 \dots & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Note that $\bar{1} = 1$. Hence $T(x) \notin [0,1]$ if $x_1 = 1$. Otherwise, since $\bar{0} = 2$ and $\bar{2} = 0$,

$$(1) \quad T(x) = \begin{cases} 0.x_2x_3x_4 \dots & \text{if } x_1 = 0, \\ 0.\bar{x}_2\bar{x}_3\bar{x}_4 \dots & \text{if } x_1 = 2. \end{cases}$$

Hence $T(x) \in [0, 1]$ if and only if $x_1 = 0$ or 2 . An easy inductive argument shows that for all $i \in \mathbb{N}$,

$$T^i(x) \in [0, 1] \quad \text{if and only if } x_i = 0 \text{ or } 2.$$

Thus $x \in \Sigma$ if and only if x is in the Cantor set and so Σ is the Cantor set. Hence inspection of the formula (1) shows that T maps Σ into itself.

The next lemma enables us to study the behavior of T on the Cantor set Σ .

LEMMA 1.

$$T^n(0.x_1x_2x_3 \dots) = \begin{cases} 0.x_{n+1}x_{n+2}x_{n+3} \dots & \text{if } x_n = 0, \\ 0.\bar{x}_{n+1}\bar{x}_{n+2}\bar{x}_{n+3} \dots & \text{if } x_n = 2. \end{cases}$$

Proof. The result holds for $n = 1$. Suppose it also holds for some $n \geq 1$. Hence

$$T^{n+1}(0.x_1x_2 \dots) = T(T^n(0.x_1x_2 \dots))$$

$$= \begin{cases} T(0.x_{n+1}x_{n+2} \dots) & \text{if } x_n = 0, \\ T(0.\bar{x}_{n+1}\bar{x}_{n+2} \dots) & \text{if } x_n = 2, \end{cases}$$

$$\begin{aligned}
 &= \begin{cases} 0.x_{n+2}x_{n+3} \dots & \text{if } (x_n = 0, x_{n+1} = 0) \text{ or } (x_n = 2, x_{n+1} = 0), \\ 0.\bar{x}_{n+2}\bar{x}_{n+3} \dots & \text{if } (x_n = 0, x_{n+1} = 2) \text{ or } (x_n = 2, x_{n+1} = 2), \end{cases} \\
 &= \begin{cases} 0.x_{n+2}x_{n+3} \dots & \text{if } x_{n+1} = 0, \\ 0.\bar{x}_{n+2}\bar{x}_{n+3} \dots & \text{if } x_{n+1} = 2. \end{cases}
 \end{aligned}$$

The result now follows by induction.

The following Theorem shows that T is chaotic in the sense of Devaney.

THEOREM 1. *The tent map T has (a) a dense set of periodic points; (b) a dense orbit; (c) sensitive dependence on initial conditions.*

Proof. (a) A point $x \in \Sigma$ of period n is one which satisfies the equation $x = T^n(x)$. In ternary, by Lemma 1, this equation is

$$0.x_1x_2x_3 \dots = \begin{cases} 0.x_{n+1}x_{n+2}x_{n+3} \dots & \text{if } x_n = 0, \\ 0.\bar{x}_{n+1}\bar{x}_{n+2}\bar{x}_{n+3} \dots & \text{if } x_n = 2. \end{cases}$$

This leads to periodic points of the form

$$(2) \quad x = 0.\underbrace{x_1x_2x_3 \dots x_{n-1}}_0 \text{ or}$$

$$(3) \quad x = 0.\underbrace{x_1x_2x_3 \dots x_{n-1}2\bar{x}_1\bar{x}_2\bar{x}_3 \dots \bar{x}_{n-1}}_0,$$

where the underbrace indicates that the block of digits is to be repeated indefinitely. To show that the set of periodic points is *dense* in Σ , let $p = 0.p_1p_2p_3 \dots \in \Sigma$. Let $\epsilon > 0$ and choose $k \in \mathbb{N}$ so that $3^{-k} < \epsilon$. Choose a periodic point $x = 0.\underbrace{p_1p_2p_3 \dots p_k}_0$. This gives $|x - p| \leq 3^{-k} < \epsilon$.

(b) Let $x = 0.A_1A_2A_3 \dots$, where each A_n is a block of digits that contains every combination of 0's and 2's of length n preceded by a zero: for example,

$$A_2 = 000\ 002\ 020\ 022.$$

Now, let $p = 0.p_1p_2p_3 \dots \in \Sigma$. Let $\epsilon > 0$ and choose $k \in \mathbb{N}$ so that $3^{-k} < \epsilon$. The finite sequence of digits $0p_1p_2p_3 \dots p_k$ occurs in x in the block A_k . Suppose this sequence starts at the m th digit of x (so that $x_m = 0$). By Lemma 1, $T^m(x) = 0.p_1p_2p_3 \dots p_k \dots$ and so in $T^m(x) - p$ the first k digits are zero, giving $|T^m(x) - p| \leq 3^{-k}$. Hence the orbit of x is *dense*.

(c) Sensitivity to initial conditions can easily be proved from first principles or by using the result of Banks et al. [1] that (a) and (b) imply (c).

Smale's horseshoe. The version of the horseshoe map H which we will analyze acts on the unit square by shrinking it three times vertically and stretching it three times horizontally, then bending it back in the unit square as shown below in Fig. 3.

We study the dynamics of the map H on the set Γ of all points (x, y) whose iterates, both forward and backward, stay in the unit square. Choosing the origin at the bottom left corner we get the following formula for H for any $(x, y) \in \Gamma$.

$$(4) \quad H(x, y) = \begin{cases} (3x, \frac{1}{3}y) & \text{if } x \in [0, \frac{1}{3}], \\ (3 - 3x, 1 - \frac{1}{3}y) & \text{if } x \in [\frac{2}{3}, 1]. \end{cases}$$

We do not give a formula for the values of H at points that map outside the unit square as these values are not needed.

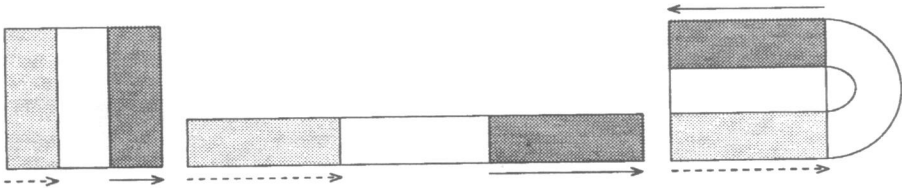


FIG. 3. The horseshoe map acting on the unit square.

The first component of $H(x, y)$ is the tent map $T(x)$. We can use the above formula to find the inverse of H . We find that $H^{-1}(x, y)$ has as its second component $T(y)$. Hence for the forward and backward iterates of (x, y) to stay in the unit square it is necessary that the ternary expansion of x and y contains only 0's and 2's. Inspection of the formula for $H(x, y)$ shows that this condition is also sufficient. Hence the set Γ of all points whose forward and backward iterates remain in the unit square is $\Sigma \times \Sigma$.

Let $y = 0.y_1y_2\dots$ in ternary. Since division by 3 shifts y one digit to the right and $1 = 0.222\dots$ in ternary,

$$H(0.x_1x_2x_3\dots, 0.y_1y_2y_3\dots) = \begin{cases} (0.x_2x_3x_4\dots, 0.0y_1y_2\dots) & \text{if } x \in [0, \frac{1}{3}], \\ (0.\bar{x}_2\bar{x}_3\bar{x}_4\dots, 0.2\bar{y}_1\bar{y}_2\dots) & \text{if } x \in [\frac{2}{3}, 1]. \end{cases}$$

Since $x_1 = 0$ if $x \in [0, \frac{1}{3}]$ and $x_1 = 2$ if $x \in [\frac{2}{3}, 1]$, we can write

$$(5) \quad H(0.x_1x_2x_3\dots, 0.y_1y_2y_3\dots) = \begin{cases} (0.x_2x_3x_4\dots, 0.x_1y_1y_2\dots) & \text{if } x_1 = 0, \\ (0.\bar{x}_2\bar{x}_3\bar{x}_4\dots, 0.x_1\bar{y}_1\bar{y}_2\dots) & \text{if } x_1 = 2. \end{cases}$$

Our aim is to generalize (5) to get an analogous formula for the iterates of H .

Geometrical motivation. Motivation for the desired generalization of (5) comes from the geometric description of the horseshoe map. We will use the geometry for the first three iterates to suggest a general formula for $H^n(x, y)$.

First iterate. Recall from Fig. 3 that H maps the two shaded vertical strips of width $1/3$ to the horizontal strips of height $1/3$. Figure 4 shows why in the formula (5), the first digit of the second component of $H(x, y)$ is x_1 .

Points with $\begin{cases} x_1=0 \\ x_1=2 \end{cases}$ map to points with $Y = \begin{cases} 0.0\dots \\ 0.2\dots \end{cases} = 0.x_1\dots$

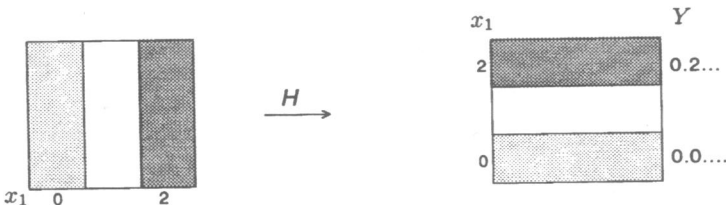


FIG. 4. Effect of $H : (x, y) \mapsto (X, Y)$ on the digits.

Using the geometrical description of the map given in Fig. 3 twice shows that H^2 acts on the vertical strips as indicated in Fig. 5.

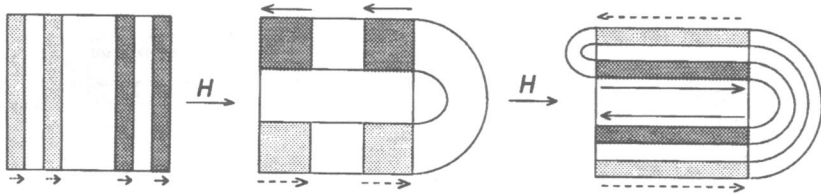


FIG. 5. H^2 maps the four vertical strips as shown. The points that map outside the unit square come from the middle thirds of the vertical strips in Fig. 4 and are therefore deleted.

We now use the geometry of the second iterate to get a diagram for H^2 analogous to Fig. 4, for H .

Second iterate. From Fig. 5, H^2 maps the four shaded vertical strips of width $1/9$ to the horizontal strips of height $1/9$. Figure 6 shows that the second digit of the second component of $H^2(x, y)$ is x_2 and how the first digit can then be obtained from x_1 .

Points with $\begin{cases} x_2=0 \\ x_2=2 \end{cases}$ map to points with the second digit of $Y = \begin{cases} 0 \\ 2 \end{cases} = x_2$;
 the first digit of Y is x_1 if $x_2 = 0$ but \bar{x}_1 if $x_2 = 2$.

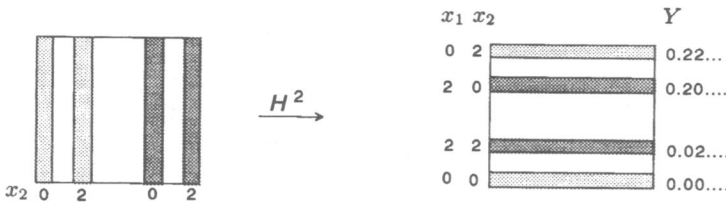


FIG. 6. Effect of $H^2 : (x, y) \mapsto (X, Y)$ on the digits.

An overall pattern seems to be emerging here. To clarify it further, we summarize the effect of H^3 on the digits.

Third iterate. Once again, restricting the image of H^3 to lie in the unit square means removing middle thirds in the previous vertical strips in the domain of H^2 . This leaves eight vertical strips in the domain of width $1/27$, which under H^3 will map to eight horizontal strips of height $1/27$. Figure 7 shows that the third digit of the second component of $H^3(x, y)$ is x_3 and how the first two digits can then be obtained from x_1 and x_2 .

Points with $\begin{cases} x_3=0 \\ x_3=2 \end{cases}$ map to points with the third digit of $Y = \begin{cases} 0 \\ 2 \end{cases} = x_3$;
 the first two digits of Y are x_2x_1 if $x_3 = 0$ but $\bar{x}_2\bar{x}_1$ if $x_3 = 2$.

Thus for $n = 1, 2$ and 3 , we have found the first n digits of Y (which is the second component of $H(x, y)$) in terms of the first n digits of x . These results, in conjunction with (5), motivate the following lemma for the iterates of the horseshoe map.

LEMMA 2.

$$H^n(x, y) = \begin{cases} (0.x_{n+1}x_{n+2} \dots, 0.x_{n-1}x_{n-2} \dots x_1x_n y_1 y_2 \dots) & \text{if } x_n = 0, \\ (0.\bar{x}_{n+1}\bar{x}_{n+2} \dots, 0.\bar{x}_{n-1}\bar{x}_{n-2} \dots \bar{x}_1\bar{x}_n \bar{y}_1 \bar{y}_2 \dots) & \text{if } x_n = 2. \end{cases}$$

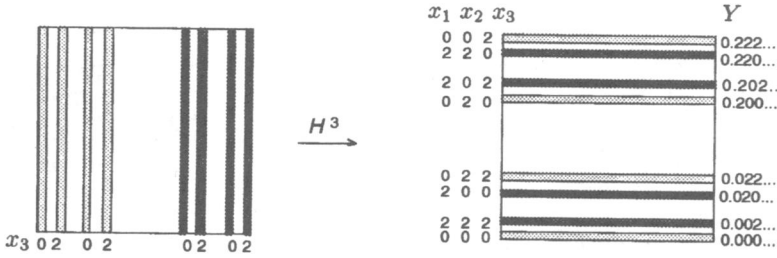


FIG. 7. Effect of $H^3 : (x, y) \mapsto (X, Y)$ on the digits.

Proof. The first component of H is the tent map T . Hence the part of the formula giving the first component of H is valid by Lemma 1. We denote the second component of H^n by S_n and so it remains to prove that

$$S_n(0.x_1x_2x_3\dots, 0.y_1y_2y_3\dots) = \begin{cases} 0.x_{n-1}x_{n-2}\dots x_1x_ny_1y_2\dots & \text{if } x_n = 0, \\ 0.\bar{x}_{n-1}\bar{x}_{n-2}\dots \bar{x}_1x_n\bar{y}_1\bar{y}_2\dots & \text{if } x_n = 2. \end{cases}$$

The result holds for $n = 1$ by definition (5) of H . Suppose it holds for $n \geq 1$. Hence

$$\begin{aligned} S_{n+1}(x, y) &= S(T^n(0.x_1x_2\dots), S_n(0.x_1x_2x_3\dots, 0.y_1y_2y_3\dots)) \\ &= \begin{cases} S(0.x_{n+1}x_{n+2}\dots, 0.x_{n-1}x_{n-2}\dots x_1x_ny_1y_2\dots) & \text{if } x_n = 0, \\ S(0.\bar{x}_{n+1}\bar{x}_{n+2}\dots, 0.\bar{x}_{n-1}\bar{x}_{n-2}\dots \bar{x}_1x_n\bar{y}_1\bar{y}_2\dots) & \text{if } x_n = 2, \end{cases} \\ &= \begin{cases} 0.x_{n+1}x_{n-1}\dots x_1x_ny_1\dots & \text{if } (x_n = 0, x_{n+1} = 0), \\ 0.x_{n+1}\bar{x}_{n-1}\dots \bar{x}_1\bar{x}_n\bar{y}_1\dots & \text{if } (x_n = 0, x_{n+1} = 2), \\ 0.\bar{x}_{n+1}\bar{x}_{n-1}\dots \bar{x}_1x_n\bar{y}_1\dots & \text{if } (x_n = 2, x_{n+1} = 2), \\ 0.\bar{x}_{n+1}x_{n-1}\dots x_1\bar{x}_ny_1\dots & \text{if } (x_n = 2, x_{n+1} = 0), \end{cases} \\ &= \begin{cases} 0.x_nx_{n-1}\dots x_1x_{n+1}y_1\dots & \text{if } x_{n+1} = 0, \\ 0.\bar{x}_n\bar{x}_{n-1}\dots \bar{x}_1x_{n+1}\bar{y}_1\dots & \text{if } x_{n+1} = 2. \end{cases} \end{aligned}$$

The result now follows by induction.

Lemma 2 allows us to show chaos for the horseshoe map in almost the same way as for the tent map.

THEOREM 2. *The horseshoe map H has (a) a dense set of periodic points; (b) a dense orbit; (c) sensitive dependence on initial conditions.*

Proof. (a) Since the action of H in the first coordinate is merely that of T , any periodic point (x, y) of H with period n must have x as in (2) or (3). Using our formula for H^n given in Lemma 2 then gives y . Hence the coordinates of the periodic points are of the form

$$\begin{aligned} x &= 0.\underbrace{x_1x_2\dots x_{n-1}}_0 \quad \text{and} \quad y = 0.\underbrace{x_{n-1}x_{n-2}\dots x_1}_0 \quad \text{or} \\ x &= 0.\underbrace{x_1x_2\dots x_{n-1}2\bar{x}_1\bar{x}_2\dots \bar{x}_{n-1}}_0 \quad \text{and} \quad y = 0.\underbrace{\bar{x}_{n-1}\bar{x}_{n-2}\dots \bar{x}_12x_{n-1}x_{n-2}\dots x_1}_0. \end{aligned}$$

To show that the set of periodic points is dense in Γ , let $(p, q) \in \Gamma$ and choose $r \in \mathbb{N}$ so that $3^{-r+1} < \epsilon$. Choose

$$x = 0.\underbrace{p_1p_2\dots p_rq_rq_{r-1}\dots q_1}_0 \quad \text{and} \quad y = 0.\underbrace{q_1q_2\dots q_r p_r p_{r-1}\dots p_1}_0.$$

The point (x, y) then has period $n = 2r + 1$. Since x agrees with p to r ternary places, and likewise y with q , it follows that $\|(x, y) - (p, q)\| \leq \sqrt{2} 3^{-r} < \epsilon$.

(b) Let $x = 0.B_2B_4B_6 \dots$ where B_{2n} contains all blocks of the form

$$p_1 p_2 \dots p_n 0 q_1 q_2 \dots q_n,$$

where the p 's and q 's are 0 or 2 (for example, $B_2 = 000\ 002\ 200\ 202$). Choose any $y \in \Sigma$. Now, given any point $(p, q) \in \Gamma$ where $p = 0.p_1 p_2 \dots$ and $q = 0.q_1 q_2 \dots$ and any $k \in \mathbb{N}$ there is a block $q_k q_{k-1} \dots q_1 0 p_1 p_2 \dots p_k$ in x where the middle zero is the n th digit of x . By Lemma 2,

$$H^n(x, y) = (0.p_1 p_2 \dots p_k \dots, 0.q_1 q_2 \dots q_k \dots)$$

and so $\|H^n(x, y) - (p, q)\| \leq \sqrt{2} 3^{-k}$. Hence the orbit of $(x, y) \in \Gamma$ is *dense*.

(c) This is the same as for Theorem 1.

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