Howard University – Spring 2020

MATH 158, Section 1

Exam 1 – practice

[12] 1. *Find the point P of intersection between the straight line $x = t$, $y = -t$, $z = t$ and the plane $3x - 2y + z = 1$, then write the equation of the straight line passing through P and perpendicular to the plane.*

Solution: By plugging the parametric equations into the plane equation, we get $3t + 2t + t = 1$, namely $t = 1/6$. The intersection pt therefore is $P = (1/6, -1/6, 1/6)$. A vector perpendicular to the plane is the one of the coefficient of the plane equation, namely $\overrightarrow{\ell} = \langle 3, -2, 1\rangle$, so the line perpendicular to the plane passing through P is

$$\begin{cases} x = 1/6 + 3t \\ y = -1/6 + 2t \\ z = 1/6 + t \end{cases}$$

[15] 2. *Find the parametric equation of the line tangent to the curve $\overrightarrow{r}(t) = 2te^t\,\overrightarrow{i} + (t+1)^2\,\overrightarrow{j} - 8\sin(2t)\,\overrightarrow{k}$ at $t = 0$.*

Solution: All you need is $\overrightarrow{r}(0)$ and $\overrightarrow{v}(0) = d\overrightarrow{r}/dt|_{t=0}$, since the equation of the tangent will be $\overrightarrow{\ell}(t) = \overrightarrow{r}(0) + t\overrightarrow{v}(0)$. Since $\overrightarrow{r}(0) = (0, 1, 0)$ and $\overrightarrow{v}(t) = \langle 2(e^t + te^t), 2(t+1), -16\cos(2t)\rangle$, so that $\overrightarrow{v}(0) = \langle 2, 2, -16\rangle$, the line is $x = 2t, y = 1 + 2t, z = -16t$.

[15] 3. *Name the following quadrics and find the lengths of the semiaxes of the conics you get by cutting the quadrics with the plane $z = 1$:*
a) $\frac{x^2}{2} + \frac{y^2}{12} - z^2 = 1$, b) $x^2 - \frac{y^2}{10} = 1$, c) $2x^2 - \frac{y^2}{4} - z^2 = 0$.

Solution: (a) is a 1-sheeted hyperboloid. Its intersection with the $z = 1$ plane is the conic (not quadric!) $\frac{x^2}{2} + \frac{y^2}{12} = 2$, namely the ellipse $\frac{x^2}{4} + \frac{y^2}{24} = 1$. Its semiaxes are 2 on the x axis and $2\sqrt{6}$ on the y axis.

(b) is a cilinder over a hyperbola. Its intersection with the $z = 1$ plane are the hyperbola $x^2 - \frac{y^2}{10} = 1$ (remark: the equation looks the deceivingly same: the first is an equation in the x, y, z 3D space, so it represents a *surface*, while the second is an equation in the x, y 2D plane, so it represents a *curve*!). Its semiaxes are respectively 1 and $\sqrt{10}$ (strictly speaking, a hyperbola has only one semiaxis, the one corresponding to the variable with the positive sign in front).

(c) is a cone. Its intersection with the $z = 1$ plane is the hyperbola $2x^2 - \frac{y^2}{4} = 1$, with semiaxes $1/\sqrt{2}$ and $1/2$.

[15] 4. *Evaluate f_{xx}, f_{xy}, f_{yx} and f_{yy} for the function $f(x, y) = e^{2x^2-3y^2}$.*

Solution: First we need to evaluate f_x and f_y:

$$f_x(x, y) = 4xe^{2x^2-3y^2}\ ,\ f_y = -6ye^{2x^2-3y^2}.$$

Hence

$$f_{xx}(x, y) = 4e^{2x^2-3y^2} + 16x^2e^{2x^2-3y^2} = 4(1 + 4x^2)e^{2x^2-3y^2}\ ,$$

$$f_{xy}(x, y) = -4x3ye^{2x^2-3y^2} = -12xye^{2x^2-3y^2}\ ,$$

$$f_{yx}(x, y) = -6y4xe^{2x^2-3y^2} = -12xye^{2x^2-3y^2} = f_{xy}(x, y)\ ,$$

$$f_{yy}(x, y) = -6e^{2x^2-3y^2} + 36y^2e^{2x^2-3y^2}. = -6(1 - 6y)e^{2x^2-3y^2}.$$

[20] 5. *Consider the curve $\vec{r}(t) = (\sqrt{2}\cos t, \sin t, \sin t)$. Find a reparametrization $\vec{\sigma}(s)$ of the curve in terms of its arc-length parameter and use it to evaluate the unit tangent vector $\vec{T}(s)$ and the curvature $k(s)$. Finally, find the coordinates of the points on the curve with $s = \pi/\sqrt{2}$ and $s = \sqrt{2}\pi$ and their reciprocal distance on the curve.*

Solution: the relation between t and the arclength parameter is given by

$$s(t) = \int_0^t \|\vec{v}(t)\|dt.$$

Since

$$\vec{v}(t) = \langle -\sqrt{2}\sin t, \cos t, \cos t\rangle$$

then

$$s(t) = \int_0^t \sqrt{2\sin^2 t + \cos^2 t + \cos^2 t}dt = \int_0^t \sqrt{2}dt = \sqrt{2}\,t.$$

Hence $t = s/\sqrt{2}$ and the curve, as function of s, writes

$$\vec{\sigma}(s) = \left(\sqrt{2}\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}\right).$$

Then

$$\vec{T}(s) = \frac{d}{ds}\vec{\sigma}(s) = \left(-\cos\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}\right)$$

and

$$k(s) = \left\|\frac{d}{ds}\vec{T}(s)\right\| = \left\|\left(-\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}, \frac{1}{2}\cos\frac{s}{\sqrt{2}}, \frac{1}{2}\cos\frac{s}{\sqrt{2}}\right)\right\| = \frac{1}{\sqrt{2}}.$$

Finally, $\vec{\sigma}(\pi/\sqrt{2}) = (0,1,1)$ and $\vec{\sigma}(\sqrt{2}\pi) = (-\sqrt{2},0,0)$. The distance of these two points on the curve is exactly $\sqrt{2}\pi - \pi/\sqrt{2}$ since s is the arclegth parameter.

[20] 6. *A point moves in cilindrical coordinates with parametric equations*

$$r = 5, \theta = 3t, z = 2t.$$

Find the corresponding parametric equations in cartesian coordinates and identify the curve.

Solution: In cartesian coordinates, the curve writes as

$$\vec{r}(t) = \langle r\cos\theta = 5\cos(3t), y = r\sin\theta = 5\sin(3t), z = 2t\rangle,$$

which is the parametric equation of a helix.

[20] 7. *The values of the first partial derivatives at $(0,1)$ of the function $z = f(x,y)$ are $f_x(0,1) = 3$ and $f_y(0,1) = -1$. Find the polar coordinates (r_0,θ_0) of $(0,1)$ and use the chain rule to find the value of the first derivatives of $g(r,\theta) = f(x(r,\theta), y(r,\theta))$ at (r_0,θ_0).*

Solution: in order to solve this problem you need to know two things: 1. the multi-variable chain rule for a function $G(t) = F(x(t), y(t))$, namely

$$\frac{d}{dt}F(t) = \dot{x}(t)F_x(x(t),y(t)) + \dot{y}(t)F_y(x(t),y(t)),$$

and the formula for polar coordinates, namely

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta\,. \end{cases}$$

Hence,

$$\frac{\partial}{\partial r} g(r,\theta) = \frac{\partial x}{\partial r} f_x(x(r,\theta), y(r,\theta)) + \frac{\partial y}{\partial r} f_y(x(r,\theta), y(r,\theta))$$

and

$$\frac{\partial}{\partial \theta} g(r,\theta) = \frac{\partial x}{\partial \theta} f_x(x(r,\theta), y(r,\theta)) + \frac{\partial y}{\partial \theta} f_y(x(r,\theta), y(r,\theta))\,.$$

BTW notice that this is actually the "full" notation. Using the "shortcut" notation I use on the blackboard, the two relations above write

$$g_r(r,\theta) = x_r(r,\theta) f_x(x(r,\theta), y(r,\theta)) + y_r(r,\theta) f_y(x(r,\theta), y(r,\theta))$$

and

$$g_\theta(r,\theta) = x_\theta(r,\theta) f_x(x(r,\theta), y(r,\theta)) + y_\theta(r,\theta) f_y(x(r,\theta), y(r,\theta))\,.$$

An even more compact notation would be

$$g_r = x_r f_x + y_r f_y$$

and

$$g_\theta = x_\theta f_x + y_\theta f_y\,,$$

which one can write matricially as

$$\begin{pmatrix} g_r \\ g_\theta \end{pmatrix} = \begin{pmatrix} x_r & y_r \\ x_\theta & y_\theta \end{pmatrix} \begin{pmatrix} f_x \\ f_y \end{pmatrix}\,,$$

but in this form you have to remember that the derivatives of x and y must be evaluated at (r,θ) while those of f at $(x(r,\theta), y(r,\theta))$.

Ok, going back to solving the problem,

$$g_r(r_0,\theta_0) = x_r(r_0,\theta_0) f_x(x(r_0,\theta_0), y(r_0,\theta_0)) + y_r(r_0,\theta_0) f_y(x(r_0,\theta_0), y(r_0,\theta_0))$$

and similarly for θ.

The problem says that (r_0, θ_0) are the polar coordinates of $(0,1)$. If you don't see at once the solution, you must solve the system

$$\begin{cases} 0 = r_0 \cos\theta_0 \\ 1 = r_0 \sin\theta_0 \end{cases}$$

from which you hopefully will be able to tell that $\theta_0 = \pi/2$ and $r_0 = 1$ (because the cosine is zero only at $\pi/2$ and $3\pi/2$, the sine is equal to 1 only at $\pi/$ and $r_0^2 = 0+1$).

Hence, finally,

$$g_r(1, \frac{\pi}{2}) =$$

$$= x_r(1, \frac{\pi}{2}) f_x\left(x(1, \frac{\pi}{2}), y(1, \frac{\pi}{2})\right) + y_r(1, \frac{\pi}{2}) f_y\left(x(1, \frac{\pi}{2}), y(1, \frac{\pi}{2})\right) =$$

$$= x_r(1, \frac{\pi}{2}) f_x(0,1) + y_r(1, \frac{\pi}{2}) f_y(0,1) =$$

$$= 3x_r(1, \frac{\pi}{2}) - y_r(1, \frac{\pi}{2}).$$

and, similarly,

$$g_\theta(1, \frac{\pi}{2}) = 3x_\theta(1, \frac{\pi}{2}) - y_\theta(1, \frac{\pi}{2}).$$

We have seen in class the derivatives of x and y w/resp to r and θ:

$$x_r = \cos\theta, y_r = \sin\theta, x_\theta = -r\sin\theta, y_\theta = r\cos\theta$$

so

$$x_r(1, \frac{\pi}{2}) = 0, y_r(1, \frac{\pi}{2}) = 1, x_\theta(1, \frac{\pi}{2}) = -1, y_\theta(1, \frac{\pi}{2}) = 0.$$

Finally, we get that

$$g_r(1, \frac{\pi}{2}) = 3 \cdot 0 - 1 = -1\,, g_\theta(1, \frac{\pi}{2}) = 3 \cdot (-1) - 0 = -3\,.$$

In matricial form,

$$\begin{pmatrix} g_r \\ g_\theta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}.$$

Extra Credit

[10] 8. *Write the equation of the plane $x = y$ in spherical coordinates.*

Solution: In spherical coordinates, $x = \rho \cos\theta \sin\phi$ and $x = \rho \sin\theta \sin\phi$, so the equation $x = y$ becomes

$$\rho \cos\theta \sin\phi = \rho \sin\theta \sin\phi \, ,$$

namely

$$\cos\theta = \sin\theta$$

and so

$$\tan\theta = 1 \, ,$$

which is also equivalent to the more explicit

$$\theta = \pi/4, 3\pi/4.$$