Howard University – Spring 2020

MATH 158, Section 1

Exam 2 – practice solutions

[20] 1. Find unitary vectors for which the directional derivative of $f(x, y) = 3xy^2 - x \ln y + \sin(\pi xy)$ is maximum, minimum and zero at $(2, 1)$. Then find the linearization of $f(x, y)$ at $(2, 1)$ and use it to evaluate $f(2.1, 1.9)$ without calculator. Finally, find the equation of the plane tangent to the graph $z = f(x, y)$ at $(2, 1)$.

Solution: Since $\vec{\nabla} f = \langle 3y^2 - \ln y + \pi y \cos(\pi xy), 6xy - x/y + \pi x \cos(\pi xy)\rangle$,

$$\vec{\nabla} f(2, 1) = \langle 3 + \pi, 10 + 2\pi\rangle.$$

The direction of maximal growth for f at $(2, 1)$ is the direction of the gradient. The unit vector in that direction is

$$\vec{u} = \frac{\langle 3 + \pi, 10 + 2\pi\rangle}{\|\langle 3 + \pi, 10 + 2\pi\rangle\|}.$$

The direction of maximal decrease is simply $-\vec{u}$. There are two unit vectors with respect to which the derivative is zero:

$$\vec{v} = \frac{\langle 10 + 2\pi, -(3 + \pi)\rangle}{\|\langle 3 + \pi, 10 + 2\pi\rangle\|} \text{ and } -\vec{v}$$

The linearization of $f(x, y)$ at $(2, 1)$ is

$$f(x, y) \simeq f(2, 1)+f_x(2, 1)(x-2)+f_y(2, 1)(y-1) = 6+(3+\pi)(x-2)+(10+2\pi)(y-1)$$

so that

$$f(2.1, 1.9) \simeq 6 + (3 + \pi) \cdot 0.1 + (10 + 2\pi) \cdot 0.9.$$

By plugging 3 for π we get $f(2, 1) \simeq 6 + 0.6 + 14.4 = 21$. An exact calculation gives $f(2.1, 1.9) \simeq 21.3637$, so with our very rough calculation we made an error of just about $0.3/21 \simeq 1\%$!

The equation of the tangent plane can be obtained by just replacing $f(x, y)$ by z in the expression of the linearization:

$$z = 6 + (3 + \pi)(x - 2) + (10 + 2\pi)(y - 1).$$

[10] 2. Find all local maxima, minima and saddle points of the function

$$f(x,y) = 11x^2 - 2x^3 + 2y^2 + 4xy$$

Solution: Local extremal points of f are given by the zeros of its gradient. Since $\vec{\nabla} f = (22x - 6x^2 + 4y, 4y + 4x)$ we must solve the system

$$\begin{cases} 22x - 6x^2 + 4y &= 0 \\ 4y + 4x &= 0 \end{cases}$$

From the second we get that $y = -x$, so the first becomes $18x - 6x^2 = 0$, whose solutions are $x = 0, 3$. Hence the gradient is zero at $(0,0)$ and $(3,-3)$. The second derivatives are $f_{xx} = 22 - 12x, f_{xy} = 4, f_{yy} = 4$, so that $f_{xx}f_{yy} - f_{xy}^2 = 88 - 48x - 16 = 72 - 48x$. At $(0,0)$, $f_{xx}f_{yy} - f_{xy}^2 = 72 > 0$ and $f_{xx} = 22 > 0$, so $(0,0)$ is a minimum. At $(3,3)$, $f_{xx}f_{yy} - f_{xy}^2 = 72 - 144 = -72 < 0$ and so it is a saddle.

[10] 3. Show that $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4}$ does not exist.

Solution: On the x axis, $\frac{x^2y^2}{x^4+3y^4} = 0$ and so the limit is 0. On the diagonal $y = x$, $\frac{x^2y^2}{x^4+3y^4} = \frac{x^2x^2}{x^4+3x^4} = \frac{x^4}{4x^4} = \frac{1}{4}$. Since we get different limits approaching $(0,0)$ on different lines, then the limit does not exist.

[20] 4. Find the absolute maximum and minimum of $f(x,y) = x^2 + 2y^2 - 4y$ in the closed disc $x^2 + y^2 \leq 9$.

Solution: The list of the candidate points to be a max or a min is the set of solutions of $\vec{\nabla} f = \vec{0}$ **inside the disc** plus the solutions of $\vec{\nabla} f = \lambda \vec{\nabla} g$ **on the boundary of the disc**.

Since $\vec{\nabla} f = \langle 2x, 4y - 4 \rangle$, its only zero is $(0,1)$. Now we need to solve the system

$$\begin{cases} 2x &= \lambda 2x \\ 4y - 4 &= \lambda 2y \\ x^2 + y^2 &= 9 \end{cases}$$

This is a 8th degree system so it can have at most 8 solutions. From the first equation we see that either $x = 0$ or $\lambda = 1$. If $x = 0$ then from the last equation we see that $y = \pm 3$, giving us the two new candidates $(0, \pm 3)$. If $\lambda = 1$ then $4y - 4 = 2y$ and so $y = 2$. Then from the last equation we get that $x = \pm\sqrt{5}$, so we have other two candidates: $(\pm\sqrt{5}, 2)$. There cannot be other candidates. Now

$$f(0,1) = -2, f(0,3) = 6, f(0,-3) = 30, f(\pm\sqrt{5}, 2) = 5$$

so the maximum is at $(0, -3)$ and the minimum at $(0, 1)$.

[20] 5. The function $z(x, y)$ is implicitly defined by the equation

$$z^5 - xz + y + 1 = 3.$$

Evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(1, 2, 1)$.

Solution: By taking the partial derivative with respect to x of both sides we find

$$5z^4 \partial_x z - z - x\partial_x z = 0$$

By taking the partial derivative with respect to y of both sides we find

$$5z^4 \partial_y z - x\partial_y z + 1 = 0$$

Hence

$$\partial_x z = \frac{z}{5z^4 - x}, \quad \partial_y z = -\frac{1}{5z^4 - x}$$

In the point $x = 1, y = 2, z = 1$ we get therefore

$$\partial_x z = \frac{1}{4}, \quad \partial_y z = -\frac{1}{4}.$$