Howard University – Spring 2020

MATH 158, Section 1

Exam 3 practice solutions

[10] 1. Use a double integral to find the volume of the solid under the plane $z = 2x + 3y$ and over the rectangle $R = \{(x, y) : 3 \le x \le 6\,,\ 1 \le y \le 2\}$.

Solution: The volume under the portion of $z = 2x + 3y$ above R is given, by definition, by

$$\iint_R (x + 3y) dA = \int_3^6 \int_1^2 (2x + 3y) dy dx = \int_3^6 \left[2xy + \frac{3}{2}y^2\right]_1^2 dy dx =$$

$$= \int_3^6 \left[2x + \frac{9}{2}\right] dx = \left[x^2 + \frac{9}{2}x\right]_3^6 = \frac{27}{2}$$

[15] 2. Evaluate the line integral $\oint_C \overrightarrow{F} \cdot d\overrightarrow{r}$ for $\overrightarrow{F} = \langle 2x - y, x + 3y - z, 3z - 2x\rangle$ over the circle of radius 1 given by the parametric equations $\overrightarrow{r}(t) = (\cos t/\sqrt{2}, \cos t/\sqrt{2}, \sin t - 1)$, $0 \le t \le 2\pi$. You might find convenient applying Stokes' Theorem.

Solution: according to Stokes' Thm, $\oint_C \overrightarrow{F} \cdot d\overrightarrow{r} = \iint_S \overrightarrow{\nabla} \times \overrightarrow{F} \cdot \overrightarrow{d}S$, where S is any surface whose boundary is C. From the fact that, in the expression of $\overrightarrow{r}(t)$, the x and y coordinates are equal for all t, we see that C is contained in the plane $x = y$ (or, equivalently, $x - y = 0$).

[REMARK: any curve in the space is planar if there some **linear** *relation among its coordinates, like $x = y$ above. An extreme case is, for example, $\overrightarrow{r}(t) = (\cos t, 5, \sin t - 1)$, in which case the curve is clearly inside $y = 5$.]*

Hence the most convenient choice for S is the planar disc bounded by C. Then $\overrightarrow{d}S = \overrightarrow{n}dA$, where $\overrightarrow{n} = \langle 1/\sqrt{2}, -1/\sqrt{2}, 0\rangle$ is the vector perpendicular to the plane $x - y = 0$ (in fact so is $\langle -1/\sqrt{2}, 1/\sqrt{2}, 0\rangle$, so by choosing this last one the integral will give you opposite sign, just do not worry about the final sign in this exercise).

On the other side,

$$\nabla\times\overrightarrow{F}=\begin{vmatrix}\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k}\\ \partial_x & \partial_y & \partial_z\\ 2x-y & x+3y-z & 3z-2x\\ & & \end{vmatrix}=-\overrightarrow{i}+2\overrightarrow{j}+2\overrightarrow{k}$$

so

$$\iint_S \overrightarrow{\nabla}\times\overrightarrow{F}\cdot\overrightarrow{d}S=\iint_S(-1/\sqrt{2}-2/\sqrt{2})dA=-3/\sqrt{2}\iint_S dA$$

The last integral is just the area of the region S, namely π, so the final value of the integral is $-3\pi/\sqrt{2}$.

[15] 3. Reverse the order of integration in

$$\int_0^2\int_{x^2}^4 xe^{-y^2}dydx$$

and evaluate the integral.

Solution: The order of integration must be inverted because the antiderivative of the function e^{-y^2} cannot be written in terms of elementary functions.

The region of integration is the region, in the first quadrant, above the parabola $y=x^2$ and below the horizontal line $y=4$. Hence, when we change their order, y ranges from 0 to 4 and, for every y, x ranges between 0 and $\sqrt{y}$.

So the integral becomes

$$\int_0^4\int_0^{\sqrt{y}} xe^{-y^2}dxdy=\int_0^4 e^{-y^2}\frac{x^2}{2}\bigg|_0^{\sqrt{y}}dxdy=\frac{1}{2}\int_0^4 ye^{-y^2}dy=$$

$$=-\frac{e^{-y^2}}{4}\bigg|_0^4=\frac{1-e^{-16}}{4}$$

(to find the antiderivative of ye^{-y^2} you have to use the substitution $u=y^2$).

[15] 4. Evaluate the double integral $\iint_R ydA$, where R is the region between the circle $x^2+y^2=9$ and $x^2+y^2=4$, in both Cartesian and Polar coordinates.

Solution:

In polar coordinates the integral writes as

$$\int_0^{2\pi}\int_2^3 (r\sin\theta) r\, dr d\theta$$

Rather than integrating first in r, it's wiser to integrate in θ. There is actually no calculation to make: if you remember the graph of the sine function you know that, over a whole period, the area below the positive part of sine is equal, in absolute value, to the one below. Their sum therefore is zero and the whole integral is zero. Otherwise just observe that

$$\int_0^{2\pi} \sin\theta d\theta = -\cos(2\pi) + \cos 0 = -1 + 1 = 0.$$

In Cartesian coordinates instead the integral must be split in four parts:

$$\int_{-3}^{-2}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} y\, dxdy + \int_{-2}^{2}\int_{\sqrt{4-x^2}}^{\sqrt{9-x^2}} y\, dxdy + \int_{-2}^{2}\int_{-\sqrt{9-x^2}}^{-\sqrt{4-x^2}} y\, dxdy + \int_{2}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} y\, dxdy =$$

$$= \int_{-3}^{-2}(9-x^2)dx + \int_{-2}^{2} 5dx - \int_{-2}^{2} 5dx + \int_{2}^{3}(9-x^2)dx = 0$$

(you can either evaluate the first and last integral and verify that their sum is zero or notice that a change of variable $u = -x$ in the first integral tranforms it in the last one but with opposite sign).

[15] 5. Consider the vector field $\overrightarrow{F}(\overrightarrow{x}) = \langle 2x - y, 3y^2 - x\rangle$.

1. Verify that $\overrightarrow{F}$ is conservative;
2. Find a function V such that $\overrightarrow{F} = \overrightarrow{\nabla} V$;
3. Evaluate the line integral of $\overrightarrow{F}$ on the curve $\overrightarrow{x}(t) = (1+\cos(\pi t), \sqrt{1+t^3})$ for $0 \le t \le 2$.

Solution: $\overrightarrow{F}$ is conservative because $\partial_x(3y^2 - x) = \partial_y(2x - y)$. To find V, we set $V(0,0) = 0$ and, in order to evaluate $V(x_0, y_0)$, we consider the parametric equations $x(t) = x_0 t$, $y(t) = y_0 t$ (that gives $(0,0)$ at $t = 0$ and (x_0, y_0) at $t = 1$) and set

$$V(x_0, y_0) = \int_0^1 \langle 2x(t) - y(t), 3y^2(t) - x(t) \rangle \cdot \langle x'(t), y'(t) \rangle dt =$$

$$= \int_0^1 \left((2x_0 t - y_0 t) x_0 + (3y_0^2 t^2 - x_0 t) y_0 \right) dt = x_0^2 - x_0 y_0 + y_0^3 \,.$$

Easy to check that, in fact, $\overrightarrow{F} = \overrightarrow{\nabla} V$.

Finally, the integral on the curve given above is simply the difference of V between the end point and the start one, namely $V(2,3) - V(2,1)$.

[15] 6. Evaluate the flux $\oiint_S \overrightarrow{F} \cdot \overrightarrow{d}S$ of $\overrightarrow{F} = \langle xyz, x + y + z, x^2 - 3z \rangle$ over the cube S given by $0 \leq x, y, z \leq 1$. You might find convenient applying the divergence theorem.

Solution: According to the divergence theorem, $\oiint_S \overrightarrow{F} \cdot \overrightarrow{d}S = \iiint_V \overrightarrow{\nabla} \cdot \overrightarrow{F} dV$, where V is the interior of the cube, namely the region $0 \leq x, y, z \leq 1$. Since $\overrightarrow{\nabla} \cdot \overrightarrow{F} = \partial_x(xyz) + \partial_y(x + y + z) + \partial_z(x^2 - 3z) = yz + 1 - 3 = yz - 2$, then

$$\oiint_S \overrightarrow{F} \cdot \overrightarrow{d}S = \int_0^1 \int_0^1 \int_0^1 (yz - 2)\, dxdydz = \int_0^1 \int_0^1 (yz - 2)\, dydz =$$

$$= \int_0^1 \left(\frac{y^2 z}{2} - 2y \right)_0^1 dz = \int_0^1 (\frac{z}{2} - 2) dz = \frac{z^2}{4} - 2z \Big|_0^1 = -\frac{7}{4}.$$

[15] 7. Write down the integral

$$\int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{x^2+y^2}^{32-x^2-y^2} dzdydx$$

in cylindrical coordinates and evaluate it.

Solution: In the xy plane the region defined by the limits is the cirle of radius 4 centered at the origin. Hence in cylindrical coordinates the integral writes as

$$\int_0^{2\pi} \int_0^4 \int_{r^2}^{32-r^2} r \, dz dr d\theta =$$

$$= \int_0^{2\pi} \int_0^4 r(32 - 2r^2) \, dr d\theta = \int_0^{2\pi} \left[16r^2 - \frac{1}{2} r^4 \right]_0^4 d\theta = 2\pi(256 - 128) = 256\pi$$

Extra Credit:

1. Evaluate the Jacobian of the transformation $x = u + v$, $y = 2u + 2v$. What does the result mean from the geometrical point of view?

Solution:

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0$$

Geometrically this means that the unit square in the uv plane becomes a segment in the xy plane, namely a parallelepiped with zero width. Indeed you can easily notice from the transformation equations that the transformed of every point (u, v) ends up on the line $y = 2x$. In particular this transformation is not bijective: for example the whole line $u + v = 1$ is mapped by the transformation in the point $(1, 2)$!

2. Find a way to evaluate

$$\int_0^\infty \int_0^\infty e^{-x^2-y^2}\, dxdy$$

Solution: This integral gets easier in polar coordinates:

$$\int_0^\infty \int_0^\infty e^{-x^2-y^2}\, dxdy = \int_0^{2\pi} \int_0^\infty e^{-r^2} r\, drd\theta = \int_0^{2\pi} -\frac{1}{2}e^{-r^2}\Big|_0^\infty d\theta =$$

$$= -\frac{1}{2}\int_0^{2\pi} \left(\lim_{r\to\infty} e^{-r^2} - e^0\right) d\theta = -\frac{1}{2}\int_0^{2\pi} (0-1) d\theta = \pi$$