On the Pointwise Limit of Complex Analytic Functions

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1. INTRODUCTION. While most elementary texts on real analysis contain an example to show that a pointwise limit of differentiable functions need not be differentiable, hardly any texts on complex analysis contain an example to show that the pointwise limit of analytic functions need not be analytic. This is a rather odd state of affairs, for the whole tenor of complex analysis is that one obtains so much from so little, and if pointwise convergence is to be sufficient one would expect it to be so in complex analysis rather than in real analysis. Our primary aim in this expository paper is to stimulate interest in examples of pointwise convergent sequences of analytic functions in the hope that they might be mentioned more often, and to reach as wide an audience as possible we have kept our discussion at an elementary level.

We begin in section 2 with two simple examples to show that the pointwise limit of a sequence of analytic functions need not be analytic. Both examples are known and, as presented here, are suitable for inclusion in a first course in complex function theory. As these examples are the main motivation for the paper, we have given them in some detail. The rest of the paper more or less follows a chronological account of the history of this topic, starting with the work of Stieltjes in 1894. It is of interest to note that despite the repeated use of Montel's theory of normal families in modern discussions of this topic, most of this work was completed before Montel introduced normal families. For more details of the history of this subject, see [**11**].

2. EXAMPLES OF A NON-ANALYTIC LIMIT FUNCTION. From a modern viewpoint, it is easy to see from Montel's theory of normal families why an example of a non-analytic pointwise limit of analytic functions requires a little work. Montel proved that the family of functions f that are analytic in a region Ω, and that map Ω into the complement of three given points w_1, w_2, and w_3 in the extended plane, is normal in Ω. It follows from this that if a sequence f_n of analytic functions is pointwise convergent in Ω, and if each f_n fails to take any of the values w_j there, then the convergence is locally uniform, and the limit is analytic. Thus, in any example f_n of the type we are seeking, $f_n(\Omega)$ must cover all but two points of the complex sphere. However, we take a different route, and we shall not use Montel's theorem at all in this paper.

Our first example of a non-analytic limit needs only Cauchy's Integral Formula; it is a slight modification of Exercise 11 in [**12**, p. 326], and a more detailed discussion (though not on this aspect) can be found in [**5**, pp. 160–162], where it is shown to be closely related to the so-called Mittag-Leffler function. The second example is, in effect, a straightforward application of a simplified (and self-contained) version of Runge's theorem applied to rational functions. The idea of this example can be found (with varying degrees of detail) in, for example, [**3**], [**8**], [**9**], [**12**], and [**14**]; we remark that Runge does not mention such an example in his fundamental paper [**13**].

Example 2.1. We shall construct an entire function E that has a radial limit 0 at ∞ in each direction except along the positive real axis $\mathbb{R}^+$, where $\mathrm{Re}[E(x)] \to +\infty$ as $x \to$

$+\infty$. Given this, we let $F(z) = \exp(-E(z))$, so that F is an entire function with radial limit 1 at ∞ in all directions except along $\mathbb{R}^+$, where it has radial limit 0. It follows that the sequence of entire functions F_n defined by $F_n(z) = F(nz)$ converges pointwise on $\mathbb{C}$ to a function that is 1 on the complement of $[0, +\infty)$ and 0 on $(0, +\infty)$.

For each positive a, let γ_a be the boundary curve of the half-strip H_a given in $\mathbb{R}^2$ by $(a, +\infty) \times (-\pi, \pi)$, and let E_a be the exterior of H_a (see Figure 1). Notice that if $a < b$ then $E_a \subset E_b$, and that $H_a \cap E_b$ is the open rectangle $(a, b) \times (-\pi, \pi)$.

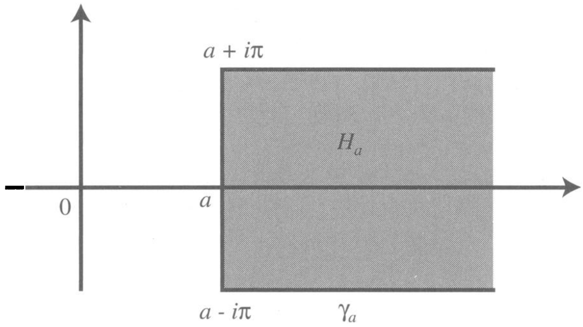

Figure 1.

Now let $f(z) = \exp(\exp(z))$. As $|f(x \pm i\pi)| = 1/\exp(\exp x)$, $|f|$ is integrable on γ_a with integral $\|f\|_{1,a}$ with respect to $|dz|$, say, and this ensures that the function

$$I_a(z) = \frac{1}{2\pi i} \int_{\gamma_a} \frac{f(w)}{w - z}\, dw$$

exists and is analytic in the complement of γ_a. Obviously,

$$|I_a(z)| \leq \frac{\|f\|_{1,a}}{2\pi \operatorname{dist}(z, \gamma_a)},$$

and this shows (i) that I_a has radial limit 0 at ∞ in all directions except along $\mathbb{R}^+$, and (ii) that I_a is bounded on the real segment $[a + 1, +\infty)$. In fact, $I_a(x) \to 0$ as $x \to +\infty$, but we do not need this.

Next, suppose that $0 < a < b$ and that $z \notin \gamma_a \cup \gamma_b$. Then

$$I_b(z) - I_a(z) = \frac{1}{2\pi i} \int_{\partial(H_a \cap E_b)} \frac{f(w)}{w - z} = f(z)\chi_{a,b}(z),$$

where $\chi_{a,b}$ is the characteristic function of the rectangle $H_a \cap E_b$. This shows that $I_b(z) = I_a(z)$ on E_a, and as $\cup_a E_a = \mathbb{C}$, this means that we can define an entire function E by $E(z) = I_a(z)$ on E_a, for each a. In particular, E has radial limit 0 at ∞ in every direction except possibly along $\mathbb{R}^+$. If $x \in H_a \cap \mathbb{R}^+$ we choose any b with $b > x$; then $x \in E_b$ and

$$E(x) = I_b(x) = I_a(x) + f(x) = O(1) + \exp(\exp x)$$

as $x \to +\infty$ so that $\operatorname{Re}[E(x)] \to +\infty$ as $x \to +\infty$. This completes the example.

We remark that (as stated in [**12**, p. 327]) the function F in this example also provides an interesting example relating to Liouville's theorem, for the function $[1 - F(z)][1 - F(-z)]$ has radial limit zero in all directions but it is not identically zero.

Example 2.2. First, we consider a compact subset K of $\mathbb{C}$ whose complement Ω is connected, and let $\mathcal{C}(K)$ be the space of functions that are defined and continuous on K equipped with the metric of uniform convergence on K. For each w in Ω let $R[w]$ be the space of finite sums of the form $\sum_n a_n(z - w)^n$, where the n here are integers. Then $R[w] \subset \mathcal{C}(K)$, and we denote the closure of $R[w]$ in $\mathcal{C}(K)$ by $\overline{R[w]}$. Note that each $R[w]$ contains all polynomials. In our view, the following lemma contains the essential idea behind Runge's theorem.

Lemma 2.3. $\overline{R[w]}$ *does not depend on* w.

Proof. The relation $w_1 \sim w_2$ if and only if $\overline{R[w_1]} = \overline{R[w_2]}$ is an equivalence relation on Ω. As Ω is connected, it suffices to show that each equivalence class is open. Take any w in Ω, and construct an open neighbourhood N of w (in Ω) such that $\operatorname{diam}(N) < \operatorname{dist}(N, K)$. Now take any w' in N. Each element of $R[w]$ is the sum of a polynomial (which lies in $\overline{R[w']}$) and a finite number of terms of the form $a_j(z - w)^{-j}$. Now

$$\frac{1}{(z - w)^m} = \frac{1}{(z - w')^m}\left(\sum_{k=0}^{\infty}\left(\frac{w - w'}{z - w'}\right)^k\right)^m.$$

The series on the right converges uniformly on K, and its partial sums lie in $R[w']$. We conclude that $\overline{R[w]} \subset \overline{R[w']}$. The reverse inclusion holds by interchanging w and w' and the proof of Lemma 2.3 is complete. ∎

Suppose now that f is any rational function all of whose poles, say $w_1, \dots, w_k$, lie in Ω. Then (by considering a partial fraction expansion of f and using Lemma 2.3) we see that

$$f \in \overline{R[w_1]} \cup \cdots \cup \overline{R[w_k]} = \overline{R[\zeta]},$$

where ζ is any chosen point in Ω. If $|\zeta|$ is chosen sufficiently large, the elements of $R[\zeta]$ are analytic in some open disc containing K and so can be uniformly approximated on K by polynomials. We deduce that *any rational function whose poles lie outside* K *can be approximated uniformly on* K *by polynomials.*

The index, or winding number, $\nu(z, \gamma)$ of a curve γ about a point z not on γ is

$$\frac{1}{2\pi i}\int_\gamma \frac{dw}{w - z},$$

and so if γ lies in Ω, then this function of z can be approximated uniformly on K by a (finite) Riemann sum

$$\sum_k \frac{a_k}{w_k - z},$$

where the w_k are on γ. It follows that *the function* $z \mapsto \nu(\gamma, z)$ *can be approximated uniformly on* K *by polynomials.* If we apply this with $K = K_n$ and $\gamma = \gamma_n$, where

$$K_n = \{0\} \cup \{z \in \mathbb{C} : 2/n \le |z| \le n,\ 1/n \le \arg(z) \le 2\pi\}$$

and γ_n, the circle $|z| = 1/n$, we obtain polynomials p_n such that $|p_n(z) - \nu(z, \gamma_n)| < 1/n$ on K_n. Clearly, $p_n(z)$ tends to 1 if $z = 0$ and to 0 otherwise.

3. THE HISTORY. Stieltjes seems to have been the first to consider pointwise convergent sequences of analytic functions, and in 1894 he proved that if a uniformly bounded sequence of analytic functions converges locally uniformly on some nonempty open subset Ω_0 of a region Ω, then the sequence converges to an analytic function throughout Ω [**15**, p. 56]. In 1901 Osgood extended this by showing that if the convergence takes place merely on a dense subset of Ω_0, then the convergence is uniform on each compact subset of Ω so that the limit is analytic throughout Ω [**8**]. Two years later, Vitali proved that it suffices to assume that convergence takes place on a set of points having a limit point in Ω [**16**].

In [**8**] Osgood considered the problem of a sequence of analytic functions converging pointwise to some function f in a region Ω, and he was well aware of the fact that f need not be analytic. Indeed he says explicitly that, by using Runge's theorem ([**13**], published in 1885), one can construct an example in which, for $n = 1, 2, 3, \ldots$, the limit function f takes the value $1/n$ on some open (nonempty) subset O_n of Ω. In the same paper, Osgood also proved the following result which, in his words, "presupposes only the bare convergence of the series—nothing more."

Osgood's Theorem. *Suppose that $f_1, f_2, \ldots$ are analytic in a region Ω, and that $f_n \to f$ pointwise in Ω. Then there is a dense open subset Ω_0 of Ω on which the convergence is locally uniform, and in which f is analytic.*

Here we are using the modern terminology that $f_n \to f$ locally uniformly in Ω if the convergence is uniform on each compact subset of Ω. Osgood's theorem does not seem to be as well known as it might be; moreover, even when it is stated the assertion that the convergence is locally uniform on Ω_0 is usually omitted. This is unfortunate, for *the set where f is analytic does not necessarily coincide with the set on which convergence is locally uniform* (see Examples 2.1 and 2.2). For a more modern reference to Osgood's theorem, see [**2**, p. 190]. More delicate questions on this topic have been discussed extensively by Lavrentieff (and others) in the 1930s, and we shall describe some of their results later in the paper.

Equicontinuity was introduced by Ascoli in 1884, and Osgood used this idea in his 1901 paper. Arzelà had shown in 1895 that equicontinuity and local uniform boundedness is equivalent to the compactness of a family of functions (although the word 'compact' was not introduced by Fréchet until 1906). All of these results, including Osgood's results, were proved before Montel introduced his theory of normal families [**7**]. Of course, Montel's ideas provide the right setting in which to discuss convergence of analytic functions, and it is well known that if $f_1, f_2, \ldots$ are analytic in a region Ω, $f_n \to f$ pointwise in Ω, and $\{f_n\}$ is a normal family in Ω, then $f_n \to f$ locally uniformly on Ω and f is analytic there. This simple result provides examples of situations in which the pointwise convergence of analytic functions does yield an analytic limit, and as an example we mention the following result.

Theorem 3.1. *Suppose that $f_1, f_2, \ldots$ are analytic and univalent in a region Ω, and that $f_n \to f$ pointwise in Ω. Then f is analytic in Ω.*

As far as we know, this result first appears in [**3**], where it is remarked that the proof (which is not given) uses Montel's deep theorem that a family of analytic maps that omits three values is a normal family. In fact, the standard growth theorem for univalent functions together with pointwise convergence shows that in this case $\{f_n\}$ is normal in Ω and so the result follows directly from Lemma 4.1. Better still, it follows from Osgood's original result as we now show.

Proof. Take any z_0 in Ω; it suffices to show that f is analytic near z_0. Using Osgood's theorem, we can find an open disk D, say $\{z : |z - \zeta| < R\}$, that lies in D, that contains z_0, and that is such that f_n converges uniformly in some neighbourhood of ζ to some analytic function g. Applying the usual growth theorem for univalent functions in the unit disc (see, for example, [**4**, Theorem 2.6]) to the function

$$F(z) = \frac{f_n(\zeta + Rz) - f_n(\zeta)}{Rf_n'(\zeta)} = z + a_2 z^2 + \cdots,$$

we see that for each n, and each w in D,

$$|f_n(w) - f_n(\zeta)| \leq \frac{R^2|f_n'(\zeta)|\,|w - \zeta|}{(R - |w - \zeta|)^2}.$$

As the sequences $f_n(\zeta)$ and $f_n'(\zeta)$ are convergent (to $g(\zeta)$ and $g'(\zeta)$, respectively), they are bounded, and this implies that the sequence (f_n) is uniformly bounded on some neighbourhood of z_0. The analyticity of f at z_0 now follows from Stieltjes's original theorem. ■

For a discussion of the pointwise limit of a sequence of Möbius transformations (whose domain need not be open) see [**1**] and [**10**].

4. OSGOOD'S THEOREM. In this section we give a proof of Osgood's theorem, and in the next section we shall look a little more closely at some of the issues raised by Osgood's paper. We begin with some convenient (but nonstandard) terminology. Suppose that $f_1, f_2, \ldots$ are analytic in a region Ω in $\mathbb{C}$; then (irrespective of whether or not (f_n) converges) we introduce the subsets $\mathcal{R}(f_n)$, $\mathcal{I}(f_n)$, $\mathcal{B}(f_n)$, $\mathcal{A}(f_n)$, and $\mathcal{S}(f_n)$ of Ω.

(a) A point z is a *regular point* of (f_n) if there is a neighbourhood of z on which (f_n) converges uniformly. If z is not a regular point of (f_n), then it is an *irregular point* of (f_n). The sets of regular and irregular points are denoted by $\mathcal{R}(f_n)$ and $\mathcal{I}(f_n)$, respectively.

(b) The sequence (f_n) is *locally bounded* at z if there exists a neighbourhood of z on which the family (f_n) is uniformly bounded. The set $\mathcal{B}(f_n)$ is the set of z at which the sequence (f_n) is locally bounded.

(c) The sequence (f_n) is *pointwise analytic* at z if there exists an open neighbourhood N of z on which the sequence $\{f_n\}$ is pointwise convergent to some function that is analytic in N. The set of points at which (f_n) is pointwise analytic is the *analytic set* $\mathcal{A}(f_n)$; the complement of $\mathcal{A}(f_n)$ in Ω is the *singular set* $\mathcal{S}(f_n)$.

It is clear that the sets $\mathcal{R}(f_n)$, $\mathcal{B}(f_n)$, and $\mathcal{A}(f_n)$ are open and that, for entirely elementary reasons,

$$\mathcal{R}(f_n) \subset \mathcal{B}(f_n) \subset \Omega, \quad \mathcal{R}(f_n) \subset \mathcal{A}(f_n) \subset \Omega.$$

The theorem of Stieltjes shows that, in the presence of pointwise convergence of (f_n) throughout Ω, we have $\mathcal{B}(f_n) \subset \mathcal{R}(f_n)$ and hence

$$\mathcal{R}(f_n) = \mathcal{B}(f_n) \subset \mathcal{A}(f_n) \subset \Omega. \tag{4.1}$$

We have already mentioned that in many cases $\mathcal{R}(f_n) \neq \mathcal{A}(f_n)$, and we shall return to discuss this shortly.

In order to prove Osgood's theorem with (4.1) available, we need show only that the pointwise convergence of the f_n in Ω implies that $\mathcal{B}(f_n)$ is dense in Ω. This does not require anything as sophisticated as the Baire Category Theorem (which is sometimes used at this point), and here we follow what is essentially Osgood's original argument.

It is enough to prove that each open disk D in Ω contains a subdisk Δ on which $\{f_n : n \geq 1\}$ is uniformly bounded. Suppose that an open disc D fails to have this property; then so does every subdisk of D. As D fails to have the property, there exists a z_1 in D, and an f_{m_1}, such that $|f_{m_1}(z_1)| > 1$. As f_{m_1} is continuous at z_1, $|f_{m_1}| > 1$ throughout some closed disc D_1 contained in the interior of D. As D fails to have the given property, so too does D_1 and using the same argument again, we can obtain a closed disc D_2 contained in the interior of D_1, and some function f_{m_2} such that $|f_{m_2}| > 2$ throughout D_2. The argument can be repeated indefinitely, and it yields a decreasing sequence of closed discs D_n, and a sequence of functions f_{m_n} such that $|f_{m_n}| \geq n$ throughout D_n. As the sequence (f_n) is unbounded at any point of the (nonempty) intersection $\bigcap_n D_n$, we have contradicted the assumption of pointwise convergence throughout D and the proof of Osgood's theorem is complete. ■

5. LAVRENTIEFF'S RESULTS. Osgood's theorem raises several issues; for example:

(a) Can any dense open subset of Ω be the set $\mathcal{A}(f_n)$ of some pointwise convergent sequence (f_n),

and

(b) can any dense open subset of Ω be the set $\mathcal{R}(f_n)$ of some pointwise convergent sequence (f_n)?

A detailed examination of these and related issues was undertaken by Lavrentieff (and others), and Lavrentieff answered both (a) and (b) by characterizing those dense open subsets of Ω that are of the form $\mathcal{A}(f_n)$, and those that are of the form $\mathcal{R}(f_n)$ (in each case, for some pointwise convergent sequence (f_n)) **[6]**. These characterizations imply that there exists a dense open subset Ω_0 of Ω that is not of the form $\mathcal{A}(f_n)$, and a dense open subset Ω_1 of Ω that is not of the form $\mathcal{R}(f_n)$. Guided by our desire for only elementary arguments, we give a simple example of each type.

Example 5.1. Suppose that the f_n are analytic and converge pointwise to f in a simply connected region Ω, and take any Jordan curve γ in $\mathcal{R}(f_n)$. Then $f_n \to f$ uniformly on γ, and the Cauchy criterion for uniform convergence combined with the Maximum Modulus Theorem shows that $f_n \to f$ uniformly on and inside γ. This proves that *if $f_n \to f$ pointwise on the simply connected region Ω, then each component of $\mathcal{R}(f_n)$ is simply connected*. It follows that if Ω_0 is obtained from Ω by removing a single point (for example), then Ω_0 cannot be the set $\mathcal{R}(f_n)$ for any sequence (f_n); thus the answer to (b) is no.

In Example 2.2 we have

$$\Omega = \mathbb{C}, \quad \mathcal{A}(p_n) = \mathbb{C}\backslash\{0\}, \quad \mathcal{R}(p_n) = \mathbb{C}\backslash[0, +\infty),$$

and in this example $\mathcal{A}(p_n) \neq \mathcal{R}(p_n)$. In general, it seems important to distinguish between $\mathcal{A}(f_n)$ and $\mathcal{R}(f_n)$, and Example 5.1 seems relevant when one tries to construct explicit examples in which the limit function is not analytic throughout Ω. Without going into details (but for reasons that are apparent from the standard proof of Runge's theorem), in order to use Runge's theorem it seems necessary to construct compact sets that omit 'thin channels' leading to ∞, and it seems to be that the uniform convergence fails in the limiting positions of these 'channels'. Indeed, Example 5.1 suggests that in order to construct a non-analytic limit it might be necessary that $\mathcal{A}(f_n) \neq \mathcal{R}(f_n)$. Interestingly, the non-analytic limit given in Example 2.1 also exhibits an 'exceptional' ray even though Runge's theorem is not used there.

We now give an example to show that not every dense open subset of Ω is of the form $\mathcal{A}(f_n)$.

Example 5.2. This example is the *Sierpiński gasket*, a fractal set that was first constructed by Sierpiński in 1916. The idea for this example (but not the use of the Sierpiński gasket) is taken from **[6]**. We begin with a brief (and informal) description of the Sierpiński gasket. Let Ω be any open disc and let v_1, v_2, and v_3 be points in Ω that form the vertices of an equilateral triangle. Let T_0 be the boundary of the triangle with vertices v_i. Now let v_3' be the midpoint of the segment $[v_1, v_2]$, and similarly for v_2' and v_3', and let T_1 be the boundary of the triangle with vertices v_1', v_2', v_3'. We now have four open equilateral triangles (each is half the size of the original triangle), and the boundary of the union of these four open triangles is $T_0 \cup T_1$. We repeat this process in each of the three outer smaller open triangles, and then in the outer triangles we obtain in this way, and so on indefinitely. The Sierpiński gasket G is the compact set $\overline{T_0 \cup T_1 \cup \cdots}$. The complement $\Omega \backslash G$ of G in Ω consists of the outside of the original triangle, together with a countable number of mutually disjoint open equilateral triangles. For us, the essential feature of G is that it has the property that any open neighbourhood of any point of G contains a complete, but 'scaled-down' copy of G.

We claim that the dense open subset $\Omega \backslash G$ of Ω *is not of the form* $\mathcal{A}(f_n)$ *for any pointwise convergent sequence of analytic functions* (f_n) *in* Ω. To see this, consider any sequence of functions f_n analytic in Ω with $f_n \to f$ pointwise on Ω and, for each z in Ω, define

$$o_n(z) = \sup\{|f_n(z) - f_m(z)| : m \geq n\}, \quad U_n = \{z \in \Omega : o_n(z) > 1\}.$$

If $z_0 \in U_n$ there is some m with $m \geq n$ and $|f_n(z_0) - f_m(z_0)| > 1$ and then, from the continuity of f_n and f_m, the same holds for all z near z_0; thus each U_n is an open set. Further, the pointwise convergence of f_n shows that $o_n \to 0$ pointwise on Ω; thus $\cap_n U_n = \emptyset$. Now Baire's Category Theorem implies (and is sometimes stated in the form) that if V_n are open subsets of a complete metric space X, and if $\cap_n V_n = \emptyset$, then there is some V_N, and some nonempty open set V in X, such that $V \cap V_N = \emptyset$. We apply this with $X = G$ and $V_n = U_n \cap G$, and deduce that there is some N and some (relatively) open set V of G such that $V \cap U_N = \emptyset$. Thus there is an open disc Δ in $\mathbb{C}$ that meets G and is such that for each z in $\Delta \cap G$,

$$\sup\{|f_N(z) - f_m(z)| : m \geq N\} \leq 1.$$

As f_N is continuous on the compact set G, it is bounded there, and we deduce that the sequence (f_n) is uniformly bounded on $\Delta \cap G$.

As every neighbourhood of every point of G contains a small closed triangular region Σ used in the construction of G, and for which $\partial\Sigma \subset G$, we deduce that there is such a closed region (in Δ) with (f_n) uniformly bounded on $\partial\Sigma$. It follows that (f_n) is uniformly bounded on the closed region Σ, and hence, from (4.1), that the interior of Σ is contained in $\mathcal{A}(f_n)$. As the interior of Σ contains points of G (because of the self-similarity of G) we deduce that some points of G are in $\mathcal{A}(f_n)$ so that, as claimed, $\Omega \backslash G \neq \mathcal{A}(f_n)$. Note that in this example we have $\Omega \backslash G \neq \mathcal{R}(f_n)$, even though each component (except the 'exterior' of T_0) of the dense open subset $\Omega \backslash G$ is simply connected.

Finally, Lavrentieff has characterized those sets that can be the singular set of some limit function, and also those sets that can be the irregular set of some sequence. To describe this characterization in a little more detail, we consider the ideas in Example 5.2. Suppose that Ω is a simply connected region and that f is the pointwise limit of the sequence (f_n) of functions analytic in Ω. Let S be the singular set of f (that is, $S = \Omega \backslash \mathcal{A}(f_n)$), and let S_0 be any closed subset of S. Then S_0 is a complete metric space and so, as for the Sierpiński gasket, there exists an open disk Δ that meets S_0 and is such that (f_n) is uniformly bounded on $\Delta \cap S_0$, and hence also on the closure $\overline{\Delta \cap S_0}$ of this set. It now follows that if D is any subregion of Δ with the property that

$$\partial D \subset \overline{\Delta \cap S_0}, \tag{5.1}$$

then the sequence (f_n) is uniformly bounded in D, and pointwise convergent there, so that f is analytic in D and hence $D \cap S = \emptyset$. To summarise, the singular set S has the property that *for every closed subset S_0 of S, there exists an open disc Δ in Ω that meets S_0 and is such that $D \cap S = \emptyset$ for any subdomain D of Δ satisfying* (5.1). Lavrentieff has shown that this is a necessary and sufficient condition for a relatively closed subset S of a simply connected region Ω to be the singular set of some limit function in Ω, and he calls any such set an M-set. He also characterizes the relatively closed subsets of a simply connected region Ω that are the irregular set of some pointwise convergent sequence (f_n) in Ω, calling such sets M^*-sets.

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Finding Those Perfect Words

RAT is perfect. Three letters, three anagrams: RAT, TAR, ART. Just perfect. CATS is perfect, too. Cats, acts, scat, and cast. Unfortunately DOGS is deficient. It has just two anagrams, including itself.

- TEA is abundant! It can make four words with only three letters. POTS is also abundant. But TEAPOSTS is prime. A word in which all of its permutations make words could be called superabundant. Can you find any?
- Most words are prime. They have no other anagrams, just themselves. Are the one-letter words prime? Perfect? Superabundant? Is there a longest prime?
- PASS is a four-letter perfect word. STAR is a four-letter abundant word. Can you find others?
- Perfect five-letter words can be tricky to find. How about that perfect STEAK? Others?
- PRIEST is a perfect six-letter word—what is the longest perfect word?
- Can you write a "perfect story"?

(Dictionary used: *Official Scrabble Players Dictionary 3*)

——Submitted by Michael Naylor
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