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Dedicated to Ivan Georgievich Petrovskii

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A survey is given of papers devoted to the problem of the existence of solutions to linear differential and pseudo-differential equations of principal type. The main results in this field are due to Lewy, Hörmander, Nirenberg, Trèves and the author. We also give a new theorem of maximal generality on local solubility of equations of principal type. By way of illustration to the exposition we mention as examples: the Lewy operator; the operator arising from the solution of the problem with directional derivatives for elliptic second order equations; non-singular operators.

Contents

Introduction

The question of the solubility (at least local) of the general linear differential equation

$$P(x, D)\,u = f(x) \tag{1}$$

has been, and remains, one of the central problems in the general theory of differential equations. As early as in 1946 Petrovskii [19] remarked "that for the simplest non-analytic equations we do not know, as a rule, whether or not there exists at least one solution. A study of this question would be of importance." Although the problem of solubility is still a long way from its final solution, results have been obtained in recent years which for differential operators with simple real characteristics take on a definitive character. We shall give an account of these results.

In this article we restrict our survey to papers known to us which deal with a single equation with a single unknown function; we refer the reader interested in systems of equations to the recent survey article by Palamodov [17]. Throughout the paper we suppose that the coefficients of (1) are infinitely differentiable with respect to all its variables.

It is, of course, local results of a negative character and positive results on global solubility that present the greatest interest in the theory. However, we cannot make these assertions without some preliminary local investigation. It would therefore seem a good idea, as a first step, to obtain a complete theory of local solubility embracing necessary and sufficient conditions.

1. It follows, of course, from the classical Cauchy-Kowalewska theorem that if the coefficients of (1) and $f(x)$ are analytic and if the principal part of the operator $P(x, D)$ of order m contains at least one pure partial derivative of order m in some direction with a non-zero coefficient, then the equation has an (analytic) solution (see [20], [23]). It is possible to widen the above class of equations with analytic coefficients: it suffices to suppose that at x_0 some derivative D^β of order m has a non-zero coefficient, where β does not belong to the convex hull of the set of multi-indices $\alpha \neq \beta$ for which $a_\alpha(x) \not\equiv 0$ in some neighbourhood of x_0 (see [23]). However, it is not possible to widen the class of the functions on the right-hand side of the equation: there exist equations with analytic coefficients that have neither classical nor generalized solutions for "most" functions $f(x)$. The first example of such an equation was discovered by Lewy in 1956 (see [23]). In §1 we give a detailed account of this equation, which has the form

$$\frac{1}{2}\left(\frac{\partial u}{\partial x}+i\frac{\partial u}{\partial y}\right)-i(x+iy)\frac{\partial u}{\partial t}=f(x, y, t).$$

2. The results obtained up to 1953 made it possible to assert the existence of solutions in the non-analytic case only for isolated classes – mainly elliptic and hyperbolic – of equations. The study of the properties of general differential operators beyond their dependence on their type and order is closely tied up with the development of the theory of distributions.

It was first shown in papers by Malgrange [35] and Ehrenpreis [27] that if (1) has constant coefficients, then the equation *is always soluble* (in any compact domain $\Omega \subset \mathbf{R}^n$) *in the class of distributions* $D'(\Omega)$ *if* $f \in D'(\Omega)$ (see also [3], [28], [42]). This remarkable theorem can be carried over to the case of variable coefficients only for very narrow classes of equations: namely for the so-called operators of constant strength (in particular, for elliptic operators) (see [23], [39]). Such operators are defined by the following condition: for any two points $x, y \in \Omega$ we have

$$\tilde{p}(x, \xi) \leqslant C_{x, y}\tilde{p}(y, \xi),$$

where

$$\tilde{p}(x, \xi) = \left(\sum \left| \frac{\partial^{|\alpha|} p(x, \xi)}{\partial \xi^\alpha} \right|^2 \right)^{1/2}.$$

In this case (1) has a solution $u \in L_2(\Omega)$ for $f \in L_2(\Omega)$, even if the coefficients are merely continuous. This last result has been generalized by Paneyakh [18] to general pseudodifferential operators whose principal part is of constant strength.

3. The article [29] by Hörmander is of fundamental significance for the theory of solubility. He showed in this article that Lewy's example is not a rare exception. *Any differential equation* (1) *is non-soluble* (*even locally*) *if at any characteristic point* $(x, \xi) \in T^*(\Omega)$ (that is, a point where $p^0(x, \xi) = 0$) *the function*

$$(2) \qquad c_1^0(x, \xi) = 2 \operatorname{Im} \sum_{j=1}^{n} \overline{\frac{\partial p^0(x, \xi)}{\partial \xi_j}} \frac{\partial p^0(x, \xi)}{\partial x_j}$$

takes a non-zero value.

This theorem was later generalized by Hörmander to pseudo-differential equations (see [31]). Hörmander's proof in [29] of the above theorem goes as follows. He shows first of all that if (1) has a solution for each $f \in D(\Omega)$, then the formal adjoint operator P^* must satisfy an *a priori* estimate of the form

$$\| v \|_s \leqslant C \| P^* v \|_t$$

for all functions $v \in D(\Omega)$. He then constructs a family of functions $v_\tau = \varphi_\tau e^{i\tau w}$ such that $\| v_\tau \|_s = 1$ and $\|P^* v_\tau\|_t \to 0$ as $\tau \to \infty$. The first step is based on an application of a theorem of Bohr and the second, namely the construction of the functions v_τ with the requisite properties, amounts to a modification of the standard WKB method, by means of which the problem reduces to that of finding solutions to the equation

$$(3) \qquad p^0(x, \operatorname{grad} w(x)) = 0, \quad w(0) = 0, \quad \operatorname{grad} w(0) = \xi$$

with a positive definite imaginary part. This method of Hörmander is the basis for all subsequent investigations on necessary conditions for solubility.

It goes without saying that this theorem does not mean that (1) is insoluble only for a single function $f(x)$. As Hörmander has shown, if in a neighbourhood ω of a point x there is a function $f(x)$ for which (1) has no solution, then the set of all such functions is of the second category (see [23]).

Furthermore, if $P(x, D)$ is a differential first order operator with homogeneous characteristic polynomial $p(x, \xi) = p^0(x, \xi)$ such that the equation $P(x, D)u = f(x)$ has no solution in any domain of $\mathbf{R}^n$, then the homogeneous equation $P(x, D)u - fu = 0$ has no non-trivial solution $u \in C^1(\Omega)$ (Lewy; see [23]). The equation with real coefficients $PP^*P^*Pu = f$, where P is the Lewy operator, has the form

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+(x^2+y^2)\frac{\partial}{\partial t}\right]^2 u+\frac{\partial^2 u}{\partial t^2}=f(x, y, t)$$

and is not soluble in any domain of $\mathbf{R}^3$ for "most" functions $f \in D(\mathbf{R}^3)$ (see Trèves [43]). In his book [23] Hörmander mentions an example of the self-adjoint second order operator with real coefficients

$$P(x, D)u=(x_2^2-x_3^2)\frac{\partial^2 u}{\partial x_1^2}+(1+x_1^2)\left(\frac{\partial^2 u}{\partial x_2^2}-\frac{\partial^2 u}{\partial x_3^2}\right)-$$
$$-x_1x_2\frac{\partial^2 u}{\partial x_1\,\partial x_2}-\frac{\partial^2 x_1x_2u}{\partial x_1\,\partial x_2}+x_1x_3\frac{\partial^2 u}{\partial x_1\,\partial x_3}+\frac{\partial^2 x_1x_3u}{\partial x_1\,\partial x_3},$$

for which (1) with some function $f \in S(\mathbf{R}^3)$ has no solution in any neighbourhood of the origin.

Also in [23] he finds very broad conditions under which (1) is always soluble if $f \in D'(\Omega)$, and he singles out the class of the so-called *essentially normal* equations, which have this property. This class is characterized by the condition: the coefficients of the equation are of class $C^1(\bar{\Omega})$ and there exists a differential operator $Q(x, D)$ of order $m - 1$ with coefficients in $C^1(\bar{\Omega})$ such that

$$c_1^0(x, \xi) = 2\mathrm{Re}\, p^0(x, \xi)\, q(x, \xi). \tag{4}$$

In particular, all operators with constant or real coefficients in the principal part (in this case $q(x, \xi) \equiv 0$) belong to this class.

4. A more complete analysis taking into account the values of the higher order derivatives at the characteristic points was carried out in [36] by Nirenberg and Trèves for the case of first order differential equations of the type

$$\sum_{j=1}^{n} a_j(x)\frac{\partial u}{\partial x_j}+a(x)u=f(x). \tag{5}$$

We say that (1) is soluble at $x_0 \in \Omega$ if there exists a neighbourhood $\omega \subset \Omega$ of this point such that $PD'(\omega) \supset C_0^\infty(\omega)$.

If the coefficients $a_j(x)$ in (5) are analytic and $\not\equiv 0$, then for this equation to be solvable at the origin *it is necessary and sufficient* that the following conditions hold. Let $p^0(x, \xi) = \sum a_j(x)\xi_j$, and let $\{\ \}$ denote the Poisson brackets

$$\{f, g\}=\sum_{j=1}^{n}\left(\frac{\partial f(x, \xi)}{\partial \xi_j}\frac{\partial g(x, \xi)}{\partial x_j}-\frac{\partial f(x, \xi)}{\partial x_j}\frac{\partial g(x, \xi)}{\partial \xi_j}\right).$$

Then the index of the first non-zero function in the list

$$p^0(x, \xi), \quad c_1(x, \xi) = \{\overline{p^0}, p^0\}, \quad c_2(x, \xi) = \{\overline{p^0}, c_1\}, \ldots$$

must be finite and even or else transfinite for all $\xi \neq 0$ and $x \in \omega$, where ω is a neighbourhood of the origin.

5. Further progress was tied up with the creation and broad application of the theory of pseudodifferential operators (see [33], [30]).

On the one hand, pseudodifferential operators naturally occur in the study of boundary value problems for differential equations (see, for example, [31]). In fact, it turns out that the solubility of the resulting pseudodifferential equation is equivalent to that of the original boundary value problem. On the other hand, the theory of pseudodifferential operators proves to be very useful in studying properties of differential equations. For instance, to investigate the solubility of equations of principal type it suffices to consider the case of pseudodifferential first order equations; this makes things considerably easier.

Let us make the notion of local solubility of a pseudodifferential operator more precise. We say that a pseudodifferential operator P is *soluble at* x_0 *if there exist two neighbourhoods* U, V *of* x_0 *such that* $U \subset V$ *and for each* $f \in C_0^\infty(U)$ *we can find a distribution* u *with support in* V *satisfying* (1) *in* U.

Hörmander's article [31] was a most important stage in the later development of the theory. He proved that *if at each characteristic point* $(x, \xi) \in T^*(\Omega)$ *the function* $c_0^1(x, \xi)$ *defined by* (2) *is negative, then* (1) *is always soluble in* Ω. Furthermore, if $f \in H_s(\Omega)$, then there exists a solution $u(x)$ of (1) in the class $H_{s+m+\frac{1}{2}}(\Omega)$ for any real s. The article also contains the above-mentioned theorem on necessary conditions for the solubility of pseudodifferential equations.

6. These results of Hörmander have been elaborated and generalized in a number of articles by the author [6] – [14], in which conditions of an algebraic character on the principal symbol $p^0(x, \xi)$ of an operator P are obtained which determine the so-called hypoelliptic operators.

These operators are defined by the existence of an *a priori* estimate of the form

$$(6) \qquad \| u \|_s \leqslant C(K, s)(\| Pu \|_{s-m+\delta} + \| u \|_{s-1}), \qquad \forall u \in C_0^\infty(K),$$

where K is an arbitrary compact subset of Ω, s is any real number, and $0 \leqslant \delta < 1$. It is not difficult to show by the Hahn-Banach theorem that if (6) holds, then the equation $P^*u = f$ is soluble, where P^* is the formal adjoint operator. If $f \in H_s(\Omega)$, then the equation $P^*u = f(x)$ has a solution $u(x)$ in the class $H_{s+m-\delta}(\Omega)$.

We now describe an algebraic condition that is equivalent to the estimate (6). Let $a_1(x, \xi) = \mathrm{Re}\, p^0(x, \xi)$, $a_2(x, \xi) = \mathrm{Im}\, p^0(x, \xi)$. For hypoelliptic operators $\mathrm{grad}_{x,\xi}\, p^0(x, \xi) \neq 0$ if $p^0(x, \xi) = 0$ (see [7]). Let

$$H_i = \sum_{j=1}^{n} \left[\frac{\partial a_i(x, \xi)}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial a_i(x, \xi)}{\partial x_j} \frac{\partial}{\partial \xi_j} \right]$$

be the operator of differentiation along the bicharacteristic of the functions $a_i(x, \xi)$ $(i = 1, 2)$, respectively. If $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\beta = (\beta_1, \ldots, \beta_m)$, where $\alpha_j \geqslant 0$, $\beta_j \geqslant 0$ are integers, then we denote by $H_1^\alpha H_2^\beta$ the operator $H_1^{\alpha_1} H_2^{\beta_1} \ldots H_1^{\alpha_m} H_1^{\beta_m}$. Let $k_1(x, \xi)$ be the least of the numbers k_1 such that $H_1^\alpha H_2^\beta a_2(x, \xi) \neq 0$ for $|\alpha + \beta| = k_1$ (if $a_1(x, \xi) + ia_2(x, \xi) \neq 0$, then we put $k_1(x, \xi) = 0$).

THEOREM 1 (see [10]). *An operator P is hypoelliptic if and only if the following two conditions are fulfilled:*

$\mathscr{A}$. *The sign of $a_1(x, \xi)$ does not change from $-$ to $+$ when moving in the positive direction along each curve $x = x(t)$, $\xi = \xi(t)$ such that*[1]

$$\frac{dx_j}{dt} = \frac{\partial a_2(x, \xi)}{\partial \xi_j}, \quad \frac{d\xi_j}{dt} = -\frac{\partial a_2(x, \xi)}{\partial x_j}, \quad a_2(x(t), \xi(t)) \equiv 0.$$

$$\mathscr{B}.\ k = \sup k_1(x, \xi) < \infty, \quad (x, \xi) \in \dot{T}^*(\Omega), \quad \xi \neq 0\ ^1).$$

Here $\delta = k/(k + 1)$. If $k_1(x_0, \xi_0) = l$, where $(x_0, \xi_0) \in \dot{T}^*(\Omega)$, $\xi_0 \neq 0$, then (6) *cannot hold for $\delta < l/(l + 1)$.*

In particular, it follows from this that there are no operators P (with smooth symbols) for which the minimum value of δ in (6) is equal to $k/(k + 1)$, where k is a non-negative integer.

In certain cases the proof of this theorem has been perfected by Trèves [47] and Eskin [24] (see also [14]).

When P is a differential operator, this theorem can be restated as follows:

THEOREM 2. *Let $P(x, D)$ be a differential operator of order m and let $k = \sup k_1(x, \xi)$ be finite. For* (1) *to be soluble it is necessary and sufficient that $k_1(x, \xi)$ takes only even values. If the latter condition holds, then* (6) *is true with $\delta = k/(k + 1)$.*

7. Nirenberg and Trèves [37] and the author [8] have simultaneously and independently obtained the most general theorems to date on necessary conditions for the solubility of (1). The author's theorem is somewhat more general in that we only suppose that at a characteristic point the vector $\mathrm{grad}_{x,\xi}\, p^0(x, \xi)$ is non-zero, while Nirenberg and Trèves suppose that $\mathrm{grad}_\xi\, p^0(x, \xi) \neq 0$. The methods of proof are also different, although, as we have already said, both proofs follow the general scheme

[1] Here and in the sequel $\dot{T}^*(\Omega)$ denotes the cotangent bundle with the zero section removed.

laid down by Hörmander. The statement of this theorem is as follows. Let $k_0(x, \xi)$ be the least number k such that $H_1^k a_2(x, \xi) \neq 0$ (if $a_1(x, \xi) + ia_2(x, \xi) \neq 0$, then we put $k_0(x, \xi) = 0$).

THEOREM 3. *Let $p^0(x^0, \xi^0) = 0$ and let $k_0(x^0, \xi^0)$ be odd, where* Re $\mathrm{grad}_{x,\xi} p^0(x^0, \xi^0) \neq 0$ *and $H_1^{k_0(x^0, \xi^0)} a_2(x^0, \xi^0) > 0$. Then the equation* (1) *is not insoluble at x_0.*

It is clear that the condition Re grad $p^0(x, \xi) \neq 0$ is no restriction because we can replace P by the operator iP.

8. New sufficient conditions for solubility of a somewhat different character have been obtained by Nirenberg and Trèves [38] and the author [11]. In [38] Nirenberg and Trèves have obtained the following result.

THEOREM 4. *Let P be a differential operator of principal type with analytic coefficients in the principal part. Suppose that for all x in a neighbourhood of x_0 the following condition holds:*

($\mathcal{P}$) *On each null characteristic of* Re $p^0(x, \xi)$ *the function* Im $p^0(x, \xi)$ *is of constant sign.*

Then there exists a neighbourhood Ω_0 of x_0 such that for each $f \in L_2(\Omega_0)$ there is a solution of (1) *of class $H_{m-1}(\Omega_0)$.*

This theorem generalizes a result by the same authors in [36]. For the case $n = 2$ this theorem was proved earlier by Trèves [48].

We replace ($\mathcal{P}$) by the condition:

($\tilde{\mathcal{P}}$) *At the points of all the null characteristics of* Re p *the function* Im p *has constant sign.*

Then the condition of analyticity can be replaced by a condition of sufficient smoothness of the coefficients.

This statement occurs also in the article [37] by Nirenberg and Trèves. Another proof of this last result and generalizations of it are given by the author in [11] (see also [13]), where sufficient conditions for the solubility of pseudodifferential equations of principal type are studied.

The proofs of the above theorems are very simple, but unfortunately the statements contain certain extra conditions which appear to be unnecessary. By refining the proof somewhat, we arrive at the following statement which is apparently best possible for the class of operators of principal type.

THEOREM 5. *Let $P(x, D)$ be a pseudodifferential operator of principal type satisfying condition ($\mathcal{A}$) of Theorem* 1. *Then for every point $x^0 \in \Omega$ and for each real s there is a neighbourhood $\omega \subset \Omega$ such that for any $f \in H_{s-m+1}(\Omega)$ with* supp $f \subset \Omega$ *there exists a function $u(x) \in H_s(\Omega)$ with compact support in Ω for which $Pu = f$ in ω and*

$$\| u \|_s \leqslant C \| f \|_{s-m+1} + C_1 \| u \|_{s-1},$$

where the constants C and C_1 do not depend on f and where C_1 tends to zero with the diameter of ω. If $s \geqslant -\frac{n}{2}$, then C_1 can be taken to be zero.

Theorem 5 remains true when P is not an operator of principal type but satisfies the following condition:

($\mathcal{P}$) *$k = \sup k_1(x, \xi) < \infty$, where the supremum is taken over all $(x, \xi) \in T^*(\Omega)$ for which $p^0(x, \xi) = 0$, $\mathrm{grad}_\xi\, p^0(x, \xi) = 0$.*

9. Clearly a different approach to the problem of solubility of (1) is possible: one can widen the class of generalized solutions under consideration by going beyond the traditional framework of distribution theory. Apparently every equation of principal type is soluble in the class of analytic functions. The class of hyperfunctions introduced by Sato (see [40]) provides a suitable apparatus for such a study. Hyperfunctions can be regarded as limiting values in the real space of holomorphic functions. However, Shapira has recently proved in [41] that the equation

$$\frac{\partial u}{\partial x_1} + ix_1 \frac{\partial u}{\partial x_2} = f(x)$$

has no solution in the class of hyperfunctions in any domain $\Omega \subset \mathbf{R}^2$, containing the origin for some function $f \in C^\infty(\mathbf{R}^2)$.

In [44] Trèves introduces the class K^s of the so-called "ultradistributions". This class K^s consists of the images under the Fourier transform of functions $\tilde{v}(\xi)$ for which

$$\int |\tilde{v}(\xi)|^2 e^{-2s|\xi|}\, d\xi < \infty.$$

In this class Trèves proves that (1) (and even the Cauchy problem for (1)) is locally soluble under very broad hypotheses.

There are a number of articles by Vishik and Eskin, Vishik and Grushin, Maz'ya and Paneyakh (see, for example, [2], [15]), where in the absence of solubility they introduce the so-called "coboundary" conditions, which salvage the position. This approach is very suitable and reasonable for the study of boundary value problems of Noether type. Unfortunately we know of hardly any results that clarify the actual behaviour of the solution in a neighbourhood of manifolds on which coboundary conditions are assigned. In this connection we mention the article of Malyutov [16] which concerns the study of the solutions of the problem with directional derivatives.

Another possible approach is to clarify the conditions on f (and not on the operator P) under which (1) has a solution in the class $D'(\Omega)$ (or smoother), that is, a description of the set $PD'(\Omega)$.

Certain results along these lines have been obtained by Hörmander in his book [23]. Any investigations in this direction should prove to be of great interest.

10. The questions of local solubility of equations not of principal type have so far been studied very little. We mention the articles of Grushin [4], Vishik and Grushin [2], Radkevich [21], [22], and Hörmander [32].

11. In the sequel we shall mention some examples illustrating the above discussion. In §1 we look at Lewy's equation and reproduce part of Lewy's original proof. In §2 we consider a fashionable problem arising from the study of the problem with directional derivatives for second order elliptic equations. This example helps us to grasp better the geometrical character of the "insoluble" equations. Finally, in §3 we prove the solubility of the so-called non-degenerate pseudodifferential equations of principal type.

§1. Lewy's example

The first example of a first order equation with smooth (analytic) coefficients having no solution with continuous first order derivatives in any domain of $\mathbf{R}^3$ was constructed by Lewy in 1957. His equation takes the form

$$Lu \equiv \frac{\partial u}{\partial \bar{z}} - iz\frac{\partial u}{\partial t} = f(x, y, t), \tag{7}$$

where $z = x + iy$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$. It seems to us that although Hörmander's proof is more suitable in the present special context, Lewy's original proof is of interest in its own right and can be useful for subsequent investigations. The main part of Lewy's proof consists of the following:

THEOREM 6. *Let $\psi(t) \in C^1$ be a real function. Suppose that the equation*

$$Lu = \psi'(t) \tag{8}$$

has a solution of class C^1 in a neighbourhood of the origin in $\mathbf{R}^3$. Then $\psi(t)$ is analytic at $t = 0$.

The general result can be deduced fairly easily from this.

PROOF (Lewy). Let $u(x, y, t)$ be a solution to (8). Let $x + iy = re^{i\theta}$ and $\rho = r^2$. Note that

$$\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} = e^{i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right). \tag{9}$$

If we write $U(t, \rho) = i\int_0^{2\pi} re^{i\theta}u(x, y, t)\,d\theta$. then we have

$$\frac{\partial}{\partial \rho}U(t, \rho) = i\int_0^{2\pi} e^{i\theta}\left[\frac{1}{2r}u + \frac{1}{2}\frac{\partial u}{\partial r}\right]d\theta.$$

But

$$\int_0^{2\pi} e^{i\theta}u\,d\theta = i\int_0^{2\pi} e^{i\theta}\frac{\partial u}{\partial \theta}\,d\theta,$$

so that

$$\frac{\partial U(t, \rho)}{\partial \rho} = \frac{i}{2} \int_0^{2\pi} e^{i\theta} \left(\frac{\partial u}{\partial r} + \frac{i}{r} \frac{\partial u}{\partial \theta} \right) d\theta,$$

and it follows from (8) and (9) that

$$\frac{\partial U(t, \rho)}{\partial \rho} = i \int_0^{2\pi} \left[\psi'(t) + ire^{i\theta} \frac{\partial u}{\partial t} \right] d\theta = 2\pi i \psi'(t) + i \frac{\partial U(t, \rho)}{\partial t}$$

or

$$\frac{\partial U}{\partial t} - i \frac{\partial U}{\partial \rho} = -2\pi\psi'(t).$$

Now $V(t, \rho) = U + 2\pi\psi(t)$ is of class C^1 and satisfies the Cauchy-Riemann equation

$$\frac{\partial V}{\partial t} - i \frac{\partial V}{\partial \rho} = 0.$$

Hence $V(t, \rho)$ is analytic in t and ρ for $\rho > 0$. Since the values of V are real when $\rho = 0$, it has an analytic continuation for values $\rho < 0$. Thus, V is analytic in t when $\rho = 0$, that is, $\psi(t)$ is analytic at $t = 0$, as required.

The fact that $u(x, y, t)$ is of class C^1 was used only once in the above proof, namely when we showed that $V(t, \rho)$ is analytic. Since even a generalized solution of the Cauchy-Riemann equation is an analytic function, it suffices to suppose that u is a bounded measurable function (or that $ru \to 0$ as $r \to 0$). Thus, if $\psi(t) \in C^\infty$ but is not analytic at $t = 0$, then (8) has no generalized solution in the class of bounded measurable functions.

We now show that (7) has no solution of class $D'(\omega)$, where ω is an arbitrary neighbourhood of the origin. By a theorem of Hörmander (see [23]) it suffices to show that the estimate

$$\| w \|_s \leqslant C \| L^* w \|_t \tag{10}$$

cannot hold for all $w \in C_0^\infty(\omega)$ for any real C, s, t. Here $L^* = -\frac{\partial}{\partial z} - i\bar{z}\frac{\partial}{\partial t}$ is the formal adjoint operator of L. The function $v = e^{-\lambda(2|z|^2+t^2-|z|^4+2it-2it|z|^2)}$ satisfies the equation $L^* v = 0$. Suppose for simplicity that $\omega = \{(x, y, t): x^2 + y^2 + t^2 \leqslant 1\}$ and that $\varphi \in C_0^\infty(\omega)$ is such that $\varphi = 1$ when $\rho^2 = x^2 + y^2 + t^2 \leqslant 1/4$. Then $f = L^*(\varphi v)$ vanishes when $\rho \leqslant 1/2$ and $\rho \geqslant 1$. It is not difficult to check that $\|L^*(\varphi v)\|_t \leqslant C\lambda^t e^{-\lambda/4}$ (here t can be taken to be a natural number. On the other hand, $\|\varphi v\|_s \geqslant c_0 \lambda^{-\frac{3}{4}+s}$, where the constant $c_0 > 0$ depends on s but not on λ. Substituting in (10) $w = \varphi v$ we obtain an inequality which cannot hold for all $\lambda > \lambda_0$. This proves that (7) is not soluble in ω.

In the general case of a pseudodifferential operator of any order a proof can be carried out along the same lines.

§ 2. The problem with directional derivatives

As a second example leading to an "insoluble" pseudodifferential equation we now consider the classical problem of Poincaré for an elliptic second order equation. This problem has been studied in articles by Bitsadze [1], Borrelli [26], Malyutov [16], the author and Kondrat'ev [5], and others.

Let $\mathbf{R}_+^{n+1}$ be the subset of $\mathbf{R}^{n+1}$ consisting of the points $X = (x_1, \ldots, x_{n+1}) = (x, x_{n+1})$ for which $x_{n+1} \geqslant 0$. Consider in $\mathbf{R}_+^{n+1}$ the equation

$$\Delta u = 0 \tag{11}$$

with the boundary condition

$$x_1^k \frac{\partial u}{\partial x_{n+1}} + a \frac{\partial u}{\partial x_1} = g(x) \quad \text{for } x_{n+1} = 0, \tag{12}$$

where $a \neq 0$ is a real constant. Note that the Shapiro-Lopatinskii condition for this problem is violated only when $x_1 = 0$. Let

$$\tilde{u}(\xi, x_{n+1}) = \int u(x, x_{n+1}) e^{-i(x, \xi)} dx$$

be the Fourier transform of $u(X)$ with respect to x. Because of (11), $\tilde{u}(\xi, x_{n+1})$ satisfies for $x_{n+1} > 0$ the equation

$$\frac{\partial^2 \tilde{u}}{\partial x_{n+1}^2} - |\xi|^2 \tilde{u} = 0,$$

the general solution of which has the form

$$\tilde{u}(\xi, x_{n+1}) = v(\xi) e^{-x_{n+1}|\xi|} + w(\xi) e^{x_{n+1}|\xi|}. \tag{13}$$

Since we are interested in a solution that is bounded for $x_{n+1} > 0$, we must put $w(\xi) = 0$. If we substitute (13) for $\tilde{u}(\xi, x_{n+1})$ in the boundary condition (12), we obtain

$$x_1^k \frac{\partial}{\partial x_{n+1}} \int v(\xi) e^{-x_{n+1}|\xi| + i(x, \xi)} d\xi + a \frac{\partial}{\partial x_1} \int v(\xi) e^{-x_{n+1}|\xi| + i(x, \xi)} d\xi = (2\pi)^n g(x)$$
$$\text{for } x_{n+1} = 0$$

that is,

$$\int (-x_1^k |\xi| + ia\xi_1) v(\xi) e^{i(x, \xi)} d\xi = (2\pi)^n g(x). \tag{14}$$

The last equation is a pseudodifferential equation with the symbol $p^0(x, \xi) = -x_1^k|\xi| + ia\xi_1$. The characteristic points $(x, \xi) \in T^*(\mathbf{R}^n)$ lie on the plane $x_1 = \xi_1 = 0$. The bicharacteristics corresponding to $\operatorname{Im} p^0(x, \xi)$ are straight lines parallel to the Ox_1-axis. It follows from Theorems 1 and 3 that (14), and therefore also our boundary value problem, are soluble for even k; when k is odd, solubility holds if and only if $a > 0$. For even k this result is in accord with Theorem 7.3 of [5]. But if k is odd, then for $a > 0$ we have a manifold of the first class in the terminology of [5]; in actual fact, the problem remains soluble even if we give supplementary values for $u(x)$ for $x_1 = 0$; in other words, the dimension of the kernel of this problem is infinite. Since the cases $a > 0$ and $a < 0$ are formally adjoint to each other, this explains the absence of *a priori* estimates for the adjoint operator for $a < 0$ and hence also the lack of a solution to (11)–(12) for any function $g(x)$. It is rather remarkable that if we give up the idea of looking for a solution in the class of distributions but allow the solution to have a singularity at $x_1 = 0$, then the problem (11)–(12) is always soluble. The most precise description of the singularities of the solution for this case has been given by Malyutov [16]. He proves that the solution so obtained is bounded and tends to a limit as $x_1 \to 0$ along each ray passing through the points of the manifold $x_1 = x_{n+1} = 0$ inside the domain.

§3. Non-degenerate operators of principal type

In this section we prove that any pseudodifferential operator P of principal type satisfying the following two conditions is soluble.

1. THE CONDITION OF NON-DEGENERACY. *At each characteristic point* $(x, \xi) \in \dot{T}^*(\Omega)$ *the vectors* $\operatorname{Re} \operatorname{grad} p^0(x, \xi)$ *and* $\operatorname{Im} \operatorname{grad} p^0(x, \xi)$ *are non-collinear.*

2. $c_1^0(x, \xi) \leqslant 0$ *if* $p^0(x, \xi) = 0$.

We show that under these conditions the following *a priori* estimate holds

$$\|u\|_{m-1} \leqslant C(\delta)\, \|Pu\|_0, \quad u \in C_0^\infty(\omega), \tag{15}$$

where $m \geqslant 0$ is the order of P, δ is the diameter of the domain ω, the constant $C(\delta)$ is independent of $u(x)$, and $C(\delta) \to 0$ as $\delta \to 0$. The solubility of (1) follows immediately from this estimate (see [23]).

The proof is based on the following propositions. Without loss of generality we may assume that $p(x, \xi) \equiv 0$ outside a small neighbourhood $\omega' \supset \omega$.

LEMMA 1 (see [7]). *If P satisfies the above conditions, then the following estimate holds*

$$\|Pu\|_0 \leqslant C(\|P^*u\|_0 + \|u\|_{m-1}). \tag{16}$$

PROOF. It is easy to see that

$$(17)\qquad \| Pu \|_0^2 = \| P^*u \|_0^2 + (C_1 u, u),$$

where $C_1 = [P^*, P] = P^*P - PP^*$ is an operator of order $2m - 1$, and the principal part of the symbol of this operator is $c_1^0(x, \xi)$. Not that if $\rho(x, \xi)$ is the distance to the characteristic manifold M, then there are positive constants a_1 and a_2 such that $a_1\rho(x, \xi) \leqslant |p(x, \xi)| \leqslant a_2\rho(x, \xi)$ for $(x, \xi) \in \dot{T}^*(\omega)$. We denote by ∇f the vector $(\mathrm{grad}_x f(x, \xi), |\xi| \mathrm{grad}_\xi f(x, \xi))$. In a small neighbourhood U of the manifold M in $\omega \times S^{n-1}$ we can find two smooth functions $a(x, \xi)$ and $b(x, \xi)$ such that on M we have the inequalities:

$$(\nabla h(x, \xi), \nabla \operatorname{Re} p^0(x, \xi)) = 0, \qquad (\nabla h(x, \xi), \nabla \operatorname{Im} p^0(x, \xi)) = 0,$$

where

$$(18)\qquad h(x, \xi) = c_1^0(x, \xi) - a(x, \xi) \operatorname{Re} p^0(x, \xi) - b(x, \xi) \operatorname{Im} p^0(x, \xi).$$

This follows from the fact that the determinant of the system

$$\Delta = |\nabla \operatorname{Re} p^0|^2 \, |\nabla \operatorname{Im} p^0|^2 - (\nabla \operatorname{Re} p^0, \nabla \operatorname{Im} p^0)^2$$

is non-zero because of condition 1. Thus, at those points of M where $c_1^0(x, \xi) = 0$ we have $h(x, \xi) = 0$, $\nabla h(x, \xi) = 0$ because the vectors $\nabla \operatorname{Re} p^0$, $\nabla \operatorname{Im} p^0$ form at each characteristic point a basis for the plane orthogonal to M. We extend the definition of $a(x, \xi)$ and $b(x, \xi)$ beyond U to obtain smooth (C^∞) functions for $(x, \xi) \in \omega \times S^{n-1}$, and then extend further to $T^*(\omega)$ as positive homogeneous functions of order $m - 1$. The function $h(x, \xi)$ is defined in $\dot{T}^*(\omega)$ by (18).

Because of the above properties of $h(x, \xi)$ and $p^0(x, \xi)$ there exists a constant $N > 0$ such that $h(x, \xi) - N|p^0(x, \xi)|^2$ is non-positive for $x \in \omega$, $|\xi| = 1$. It follows from (17) that for all functions u in $C_0^\infty(\omega)$

$$\| Pu \|_0^2 = \| P^*u \|_0^2 + (Su, u) + (RP^*u, u) + (Tu, u),$$

where S is a pseudodifferential operator of order $2m - 1$ with the symbol $h(x, \xi) - N|p^0(x, \xi)|^2 |\xi|^{-1}$, the operator R has order $m - 1$, and T is of order $2m - 2$.

It follows from Gårding's inequality (see [31]) that

$$(Su, u) \leqslant C \| u \|_{m-1}^2, \qquad u \in C_0^\infty(K).$$

Hence

$$\| Pu \|_0^2 \leqslant C_1 (\| P^*u \|_0^2 + \| u \|_{m-1}^2),$$

as required.

LEMMA 2 (see [23], [18], [45]). *If $P(x, D)$ is an operator of* *principal type, then*

(19) $$\| u \|_{m-1} \leqslant C(\delta)(\| P^*u \|_0 + \| Pu \|_0), \quad u \in C_0^\infty(\omega),$$

where δ *is the diameter of* ω, *the constant* $C(\delta)$ *does not depend on* $u(x)$, *and* $C(\delta) \to 0$ *as* $\delta \to 0$.

PROOF. Let $\epsilon > 0$ be any (sufficiently small) number. Let $\psi(x) \in C_0^\infty(\mathbf{R}^n)$, where $\psi(x) = 1$ when $|x| < \epsilon$ and $\psi(x) = 0$ when $|x| > 2\epsilon$. Suppose for simplicity that the domain ω lies in the ball $|x| \leqslant \delta$. It suffices to prove (19) for operators $P(x, D)$ with homogeneous symbol $p^0(x, \xi)$, because if $P = P_0 + Q$, where Q is an operator of order $m - 1$, then

$$\| P_0 u \|_0 + \| P_0^* u \|_0 \leqslant \| Pu \|_0 + \| P^*u \|_0 + C_1 \| u \|_{m-1}.$$

If δ is so small that $C_1 C(\delta) < 1$, then from the inequality

$$\| u \|_{m-1} \leqslant \frac{1}{2} C(\delta)(\| P_0 u \|_0 + \| P_0^* u \|_0)$$

we immediately obtain (19). Thus, we may suppose from now on that $P = P_0$. Denote by $P^{(j)}(x, D)$, $P^{(jj)}(x, D)$ the pseudodifferential operators with the symbols $\partial p^0(x, \xi)/\partial \xi_j$, $\partial^2 p^0(x, \xi)/\partial \xi_j^2$, respectively. Note that

$$P^{(j)}(x, D) u = P^{(j)}(x, D)(\psi u) = P(x_j \psi u) - x_j P u + Tu,$$

where T is an operator of order $-\infty$.

Therefore

$$\begin{aligned} \| P^{(j)}(x, D) u \|_0^2 &= (P(x_j\psi u), P^{(j)}u) - (x_j\psi Pu, P^{(j)}u) + (Tu, P^{(j)}u) = \\ &= (P^{(j)*}(x_j\psi u), P^*u) + (x_j\psi u, [P^*, P^{(j)}]u) - (x_j\psi Pu, P^{(j)}u) + \\ &+ (Tu, P^{(j)}u) = (P^{(jj)*}u, P^*u) + (x_j\psi P^{(j)*}u, P^*u) + \\ &+ (x_j u, [P^*, P^{(j)}]u) - (x_j\psi Pu, P^{(j)}u) + (T_1 u, P^{(j)}u), \end{aligned}$$

where T_1 is an operator of order $-\infty$. Hence we see that

(20) $$\begin{aligned} \| P^{(j)}(x, D) u \|_0^2 &\leqslant C[\| u \|_{m-2} \| P^*u \|_0 + \varepsilon \| u \|_{m-1} \| P^*u \|_0 + \\ &+ \varepsilon \| u \|_{m-1}^2 + \varepsilon \| Pu \| \| u \|_{m-1}] + C(\varepsilon) \| u \|_{m-2}^2 \leqslant \\ &\leqslant C'\varepsilon(\| Pu \|_0 + \| P^*u \|_0) \| u \|_{m-1} + C\varepsilon \| u \|_{m-1}^2 + C(\varepsilon) \| u \|_{m-2}^2. \end{aligned}$$

Here we have used the facts that $|x_j| \leqslant \epsilon$ in supp $\psi(x)$ and that $\| u \|_{m-2} \leqslant \delta \| u \|_{m-1}$ for $u \in C_0^\infty(\omega)$. It is clear that $P^{(j)}u = P^{(j)}\psi u = \psi P^{(j)}u + T_2 u$, where the order of T_2 is $-\infty$, so that

(21) $$\| \psi P^{(j)}u \|_0^2 \leqslant \| P^{(j)}u \|_0^2 + C_1(\varepsilon) \| u \|_{m-2}^2.$$

Further, because $p^{0(j)}(x, \xi)$ is continuous, we have

$$\| \psi [P^{(j)}(x, D) - P^{(j)}(0, D)] u \|_0^2 \leqslant C_1 \varepsilon \| u \|_{m-1}^2 + C_2 \| u \|_{m-2}^2,$$

so that

(22) $$\|\psi P^{(j)}(0, D)\, u\|_0^2 \leqslant \|\psi P^{(j)}(x, D)\, u\|_0^2 + C_1\varepsilon\, \|u\|_{m-1}^2 + C_2\, \|u\|_{m-2}^2.$$

Since P is an operator of principal type,

$$\sum_{j=1}^{n} \left| \frac{\partial p^0(0, \xi)}{\partial \xi_j} \right|^2 \geqslant c_0 |\xi|^{2(m-1)} \geqslant \frac{c_0}{2}(1 + |\xi|^2)^{m-1} - C_3 (1 + |\xi|^2)^{m-2},$$

where $c_0 > 0$. Hence from Parseval's equality we obtain

(23) $$\sum_{j=1}^{n} \|P^{(j)}(0, D)\, u\|_0^2 \geqslant \frac{c_0}{2} \|u\|_{m-1}^2 - C_3 \|u_{m-2}^2.$$

As before (see (21)), we can show that

(24) $$\|P^{(j)}(0, D)\, u\|_0^2 \leqslant \|\psi P^{(j)}(0, D)\, u\|_0^2 + C_2(\varepsilon)\, \|u\|_{m-2}^2.$$

Finally, combining (20)–(24), we obtain

$$\frac{c_0}{2} \|u\|_{m-1}^2 \leqslant C'\varepsilon\, (\|Pu\|_0 + \|P^*u\|_0)\, \|u\|_{m-1} + (C + C_1)\, \varepsilon\, \|u\|_{m-1}^2 + C_3(\varepsilon)\, \|u\|_{m-2}^2.$$

Since $u \in C_0^\infty(\omega)$, we have $\|u\|_{m-2} \leqslant \delta\, \|u\|_{m-1}$. We can choose ϵ and then δ sufficiently small so that $(C + C_1)\, \epsilon < c_0/8$, $\delta^2 C_3(\epsilon) < c_0/8$, and then

$$\frac{c_0}{4} \|u\|_{m-1}^2 \leqslant C'\varepsilon\, (\|Pu\|_0 + \|P^*u\|_0)\, \|u\|_{m-1},$$

hence cancelling $\|u\|_{m-1}$ we obtain (19).

PROOF OF THE INEQUALITY (15). Combining (16) and (19) we see that for $u \in C_0^\infty(\omega)$

$$\|u\|_{m-1} \leqslant C(\delta)\, (1 + C)\, \|P^*u\|_0 + C(\delta)\, C\, \|u\|_{m-1}.$$

If δ is so small that $C(\delta)\, C < 1/2$, then it follows that $\|u\|_{m-1} \leqslant 2C(\delta)\,(1 + C)\, \|P^*u\|_0$, as required.

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