1.3 Vector Fields and Flows.

This section introduces vector fields on Euclidean space and the flows they determine. This topic puts together and globalizes two basic ideas learned in undergraduate mathematics: the study of vector fields on the one hand and differential equations on the other.

Definition 1.3.1. *Let $r \geq 0$ be an integer. A C^r* ***vector field*** *on $\mathbb{R}^n$ is a mapping $X : U \to \mathbb{R}^n$ of class C^r from an open set $U \subset \mathbb{R}^n$ to $\mathbb{R}^n$. The set of all C^r vector fields on U is denoted by $\mathfrak{X}^r(U)$ and the C^∞ vector fields by $\mathfrak{X}^\infty(U)$ or $\mathfrak{X}(U)$.*

We think of a vector field as assignning to each point $x \in U$ a vector $X(x)$ based (i.e., bound) at that same point.

Example. Consider the force field determined by Newton's law of gravitation. Here the set U is $\mathbb{R}^3$ minus the origin and the vector field is defined by

$$\mathbf{F}(x, y, x) = -\frac{mMG}{r^3}\mathbf{r},$$

where m is the mass of a test body, M is the mass of the central body, G is the constant of gravitation, $\mathbf{r}$ is the vector from the origin to (x, y, z), and $r = (x^2 + y^2 + z^2)^{1/2}$; see Figure 1.3.1.

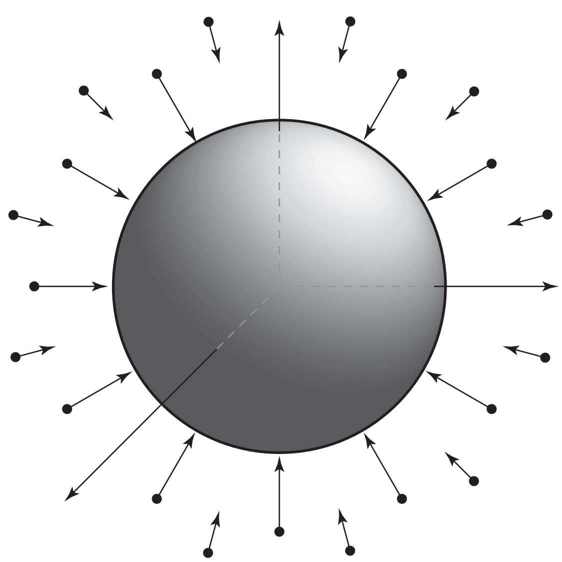

FIGURE 1.3.1. The gravitational force field.

♦

Consider a general physical system that is capable of assuming various "states" described by points in a set Z. For example, Z might be $\mathbb{R}^3 \times \mathbb{R}^3$ and a state might be the position and momentum $(\mathbf{q}, \mathbf{p})$ of a particle. As time passes, the state evolves. If the state is $z_0 \in Z$ at time t_0 and this changes to z at a later time t, we set

$$F_{t,t_0}(z_0) = z$$

and call F_{t,t_0} the ***evolution operator***; it maps a state at time t_0 to what the state would be at time t; i.e., after time $t - t_0$ has elapsed. "Determinism" is expressed by the law

$$F_{t_2,t_1} \circ F_{t_1,t_0} = F_{t_2,t_0} \quad F_{t,t} = \text{ identity},$$

sometimes called the ***Chapman-Kolmogorov law***.

The evolution laws are called ***time independent*** when F_{t,t_0} depends only on $t - t_0$; i.e.,

$$F_{t,t_0} = F_{s,s_0} \quad \text{if} \quad t - t_0 = s - s_0.$$

Setting $F_t = F_{t,0}$, the preceding law becomes the ***group property***:

$$F_\tau \circ F_t = F_{\tau+t}, \quad F_0 = \text{ identity}.$$

We call such an F_t a ***flow*** and F_{t,t_0} a ***time-dependent flow***, or an evolution operator. If the system is defined only for $t \geq 0$, we speak of a ***semi-flow***.

It is usually not F_{t,t_0} that is given, but rather the ***laws of motion***. In other words, differential equations are given that we must solve to find the flow. In general, Z is a manifold (a generalization of a smooth surface), but we confine ourselves here to the case that $Z = U$ is an open set in some Euclidean space $\mathbb{R}^n$. These equations of motion have the form

$$\frac{dx}{dt} = X(x), \quad x(0) = x_0$$

where X is a (possibly time-dependent) vector field on U.

Example. The motion of a particle of mass m under the influence of the gravitational force field is determined by Newton's second law:

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F};$$

i.e., by the ordinary differentatial equations

$$m\frac{d^2x}{dt^2} = -\frac{mMGx}{r^3};$$
$$m\frac{d^2y}{dt^2} = -\frac{mMGy}{r^3};$$
$$m\frac{d^2z}{dt^2} = -\frac{mMGz}{r^3};$$

Letting $\mathbf{q} = (x, y, z)$ denote the position and $\mathbf{p} = m(d\mathbf{r}/dt)$ the momentum, these equations become

$$\frac{d\mathbf{q}}{dt} = \frac{\mathbf{p}}{m}; \quad \frac{d\mathbf{p}}{dt} = \mathbf{F}(\mathbf{q})$$

The phase space here is the open set $U = (\mathbb{R}^3 \backslash \{\mathbf{0}\}) \times \mathbb{R}^3$. The right-hand side of the preceding equations define a vector field by

$$X(\mathbf{q}, \mathbf{p}) = ((\mathbf{q}, \mathbf{p}), (\mathbf{p}/m, \mathbf{F}(\mathbf{q}))).$$

In courses on mechanics or differential equations, it is shown how to integrate these equations explicitly, producing trajectories, which are planar conic sections. These trajectories comprise the flow of the vector field. ♦

Relative to a chosen set of Euclidean coordinates, we can identify a vector field X with an n-component vector function $(X^1(x), \ldots, X^n(x))$, the components of X.

Definition 1.3.2. *Let $U \subset \mathbb{R}^n$ be an open set and $X \in \mathfrak{X}^r(U)$ a vector field on U. An **integral curve** of X with initial condition x_0 is a differentiable curve c defined on some open interval $I \subset \mathbb{R}$ containing 0 such that $c(0) = x_0$ and $c'(t) = X(c(t))$ for each $t \in I$.*

Clearly c is an integral curve of X when the following system of ordinary differential equations is satisfied:

$$\begin{array}{rcl} \dfrac{dc^1}{dt}(t) & = & X^1(c^1(t), \ldots, c^n(t)) \\ \vdots & & \vdots \\ \dfrac{dc^n}{dt}(t) & = & X^n(c^1(t), \ldots, c^n(t)) \end{array}$$

We shall often write $x(t) = c(t)$, an admitted abuse of notation. The preceding system of equations are called ***autonomous***, when X is time independent. If X were time dependent, time t would appear explicitly on the right-hand side. As we have already seen, the preceding system of equations includes equations of higher order by the usual reduction to first-order systems.

Theorem 1.3.3 (Local Existence, Uniqueness, and Smoothness). *Suppose that $U \subset \mathbb{R}^n$ is open and that X is a C^r vector field on U for some $r \geq 1$. For each $x_0 \in U$, there is a curve $c : I \to U$ with $c(0) = x_0$ such that $c'(t) = X(c(t))$ for all $t \in I$. Any two such curves are equal on the intersection of their domains. Furthermore, there is a neighborhood U_0 of the point $x_0 \in U$, a real number $a > 0$, and a C^r mapping $F : U_0 \times I \to U$, where I is the open interval $]-a, a[$, such that the curve $c_u : I \to U$, defined by $c_u(t) = F(u, t)$ is a curve satisfying $c_u(0) = u$ and the differential equations $c_u'(t) = X(c_u(t))$ for all $t \in I$.*

This theorem has many variants. We refer to Coddington and Levinson [1955] and Hartman [2002] for a thorough discussion of most of them. For example, with just continuity of X one can get existence (the Peano existence theorem) without uniqueness. The equation in one dimension given by $\dot{x} = \sqrt{x}, x(0) = 0$ has the two C^1 solutions $x_1(t) = 0$ and $x_2(t)$ which is defined to be 0 for $t \leq 0$ and $x_2(t) = t^2/4$ for $t > 0$. This shows that one can indeed have existence without uniqueness for continuous vector fields.

The proof of the preceding theorem is based on the following.

Theorem 1.3.4 (**Local Existence and Uniqueness**). *Let $U \subset \mathbb{R}^n$ be an open set, and $X : U \to \mathbb{R}^n$ be a Lipschitz map; i.e., there is a constant $K > 0$ such that*

$$\|X(x) - X(y)\| \leq K\|x - y\|$$

for all $x, y \in U$. Let $x_0 \in U$ and suppose the closed ball of radius b, $B_b(x_0) = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq b\}$ lies in U, and that $\|X(x)\| \leq M$ for a constant M and all $x \in B_b(x_0)$. Let $t_0 \in \mathbb{R}$ and let $\alpha = b/M$. Then there is a unique C^1 curve $x(t)$, $t \in [t_0 - \alpha, t_0 + \alpha]$ such that

$$x(t) \in B_b(x_0) \quad \textit{and} \quad \begin{cases} x'(t) = X(x(t)) \\ x(t_0) = x_0. \end{cases}$$

Proof. The conditions $x'(t) = X(x(t)), x(t_0) = x_0$ are equivalent to the integral equation

$$x(t) = x_0 + \int_0^t X(x(s))\, ds \tag{1.3.1}$$

Put $x_0(t) = x_0$ and define inductively

$$x_{n+1}(t) = x_0 + \int_{t_0}^t X(x_n(s))\, ds.$$

This process is called ***Picard iteration***. Clearly $x_n(t) \in B_b(x_0)$ for all n and $t \in [t_0 - \alpha, t_0 + \alpha]$ by definition of α. We claim that

$$\|x_{n+1}(t) - x_n(t)\| \leq \frac{MK^n|t - t_0|^{n+1}}{(n+1)!}. \tag{1.3.2}$$

To see this, we proceed by induction. For $n = 0$ this reads

$$\|x_1(t) - x_0\| \leq M|t - t_0|$$

However, this follows since by definition of $x_1(t)$:

$$x_1(t) - x_0 = \int_{t_0}^t X(x_0)\, ds$$

and since $\|X(x_0)\| \leq M$. Now assume that equation (1.3.2) holds for n. To prove it for $n+1$, we estimate as follows:

$$\begin{aligned}
\|x_{n+2}(t) - x_{n+1}(t)\| &= \left\| \int_{t_0}^{t} \left(X(x_{n+1}(s)) - X(x_n(s))\right) ds \right\| \\
&\leq \int_{t_0}^{t} \left(K\|x_{n+1}(s) - x_n(s)\|\right) ds \\
&\leq \int_{t_0}^{t} \left(K \frac{MK^n|t-t_0|^{n+1}}{(n+1)!}\right) ds \\
&\leq \frac{MK^{n+1}|t-t_0|^{n+2}}{(n+2)!}
\end{aligned}$$

and so we have proved equation (1.3.2) for $n+1$ as required. Since

$$\frac{MK^n}{(n+1)!}(t-t_0)^{n+1} \leq \frac{MK^n}{(n+1)!}\alpha^{n+1}$$

and the series with these quantities as terms is convergent, we see, writing $\|x_{n+p} - x_n\|$ as a telescoping sum, that the functions $x_n(t)$ form a uniformly Cauchy sequence and hence converge uniformly to a continuous function $x(t)$. This curve $x(t)$ satisfies the integral equation (1.3.1). Since $x(t)$ is continuous, the integral equation in fact shows that it is C^1 from the fundamental theorem of Calculus. This proves existence.

For uniqueness, let $y(t)$ be another solution. By induction we find that $\|x_n(t) - y(t)\| \leq MK^n|t-t_0|^{n+1}/(n+1)!$; thus, letting $n \to \infty$ gives $x(t) = y(t)$. ∎

A local existence result may alternatively be proven by making use of the contraction mapping principle, at least if we "give a little" and assume that α also satisfies $\alpha < 1/K$. The idea is to consider the space $\mathcal{C}$ of continuous curves $c(t)$ defined on the closed interval $[t_0-\alpha, t_0+\alpha]$. These curves should satisfy $c(t_0) = x_0$ and lie in a closed ball of radius b. We consider $\mathcal{C}$ as a complete metric space with the distance function given by the supremum of the distance between two curves. Now we set up a mapping

$$\mathcal{F} : \mathcal{C} \to \mathcal{C}$$

defined by

$$\mathcal{F}(c(\cdot))(t) = x_0 + \int_{t_0}^{t} X(c(s))\, ds$$

Then one checks, using the fact that α is less than or equal to b/M that $\mathcal{F}$ maps $\mathcal{C}$ to itself, and the Lipschitz condition on X provides an estimate on $\mathcal{F}(c_1(\cdot)) - \mathcal{F}(c_2(\cdot))$ in the sup norm. The condition that α is less than $1/K$ shows that $\mathcal{F}$ is a contraction. The unique fixed point is the desired integral curve. *Notice, however, that the time of existence given by this technique is not as good as what was given in the local existence result, Theorem* 1.3.4.

Dependence on Parameters and Time. The same argument holds if X depends explicitly on t and/or on a parameter ρ, is jointly continuous in (t, ρ, x), and is Lipschitz in x uniformly in t and ρ. Since $x_n(t)$ is continuous in (x_0, t_0, ρ) so is $x(t)$, being a uniform limit of continuous functions; thus the integral curve is jointly continuous in (x_0, t_0, ρ). [3]

The following inequality is of basic importance in not only existence and uniqueness theorems, but also in making estimates on solutions.

Theorem 1.3.5 (Gronwall's Inequality). *Let $f, g : [a, b[\to \mathbb{R}$ be continuous and nonnegative. Suppose there is a constant $A \geq 0$ such that for all t satisfying $a \leq t \leq b$,*

$$f(t) \leq A + \int_a^t f(s)\, g(s)\, ds.$$

Then

$$f(t) \leq A \exp\left(\int_a^t g(s)\, ds\right) \quad \text{for all} \quad t \in [a, b[.$$

Proof. First suppose $A > 0$. Let

$$h(t) = A + \int_a^t f(s)\, g(s)\, ds;$$

thus $h(t) > 0$. Then $h'(t) = f(t)g(t) \leq h(t)g(t)$. Thus $h'(t)/h(t) \leq g(t)$. Integration gives

$$h(t) \leq A \exp\left(\int_a^t g(s)\, ds\right).$$

This gives the result for $A > 0$. If $A = 0$, then we get the result by replacing A by $\varepsilon > 0$ for every $\varepsilon > 0$; thus h and hence f is zero. ■

Lemma 1.3.6. *Let X be as in Theorem 1.3.4. Let $F_t(x_0)$ denote the solution (= integral curve) of $x'(t) = X(x(t)), x(0) = x_0$. Then there is a neighborhood V of x_0 and a number $\varepsilon > 0$ such that for every $y \in V$ there is a unique integral curve $x(t) = F_t(y)$ satisfying $x'(t) = X(x(t))$ for all $t \in [-\varepsilon, \varepsilon]$, and $x(0) = y$. Moreover,*

$$\|F_t(x) - F_t(y)\| \leq e^{K|t|}\|x - y\|.$$

Proof. Choose $V = B_{b/2}(x_0)$ and $\varepsilon = b/2M$. Fix an arbitrary $y \in V$. Then $B_{b/2}(y) \subset B_b(x_0)$ and hence $\|X(z)\| \leq M$ for all $z \in B_{b/2}(y)$. By

[3] The reader who is familiar with Banach spaces will notice that this proof works essentially unchanged in that context with $\mathbb{R}^n$ replaced by a Banach space. However, one has to be cautious not to think that this will always enable one to deal with partial differential equations (such as dealing with equations like the heat equation ($\partial u/\partial t = \nabla^2 u$) by just choosing the Banach space to be a space of functions.

Theorem 1.3.4 with x_0 replaced by y, b by $b/2$, and t_0 by 0, there exists an integral curve $x(t)$ of $x'(t) = X(x(t))$ for $t \in [-\varepsilon, \varepsilon]$ and satisfying $x(0) = y$. This proves the first part. For the second, let $f(t) = \|F_t(x) - F_t(y)\|$. Clearly,

$$\begin{aligned} f(t) &= \left\| \int_0^t [X(F_s(x)) - X(F_s(y))]\, ds + x - y \right\| \\ &\leq \|x - y\| + K \int_0^t f(s)\, ds, \end{aligned}$$

so the result follows from Gronwall's inequality. ■

This result shows that $F_t(x)$ depends in a continuous, indeed Lipschitz, manner on the initial condition x and is jointly continuous in (t, x). Again, the same result holds if X depends explicitly on t and on a parameter ρ is jointly continuous in (t, ρ, x), and is Lipschitz in x uniformly in t and ρ. We let $F^\rho_{t,\lambda}(x)$ be the unique integral curve $x(t)$ satisfying $x'(t) = X(x(t), t, \rho)$ and $x(\lambda) = x$. By the remarks following Theorem 1.3.4, $F^\rho_{t,t_0}(x)$ is jointly continuous in the variables (t_0, t, ρ, x), and is Lipschitz in x, uniformly in (t_0, t, ρ).

We now want to work towards showing that F_t is C^r if X is. We will do this first locally and later will also show this for all t for which the flow is defined.

For the next lemma, recall that by the mean value theorem, a C^1-function is locally Lipschitz.

Lemma 1.3.7. *Let X in Theorem 1.3.4 be of class C^k, where $1 \leq k \leq \infty$, and let $F_t(x)$ be defined as before. Then locally in $(t, x), F_t(x)$ is of class C^k in x and is C^{k+1} in the t-variable.*

Proof. We define $\psi(t, x)$ taking values in the vector space $L(\mathbb{R}^n, \mathbb{R}^n)$ consisting of the set of linear maps of $\mathbb{R}^n$ to $\mathbb{R}^n$ (this space is isomorphic to $\mathbb{R}^{n^2}$), to be the solution of the "linearized" or "first variation" equations:

$$\frac{d}{dt}\psi(t, x) = \mathbf{D}X(F_t(x)) \circ \psi(t, x),$$

with $\psi(0, x) =$ identity, where $\mathbf{D}X(y) : \mathbb{R}^n \to \mathbb{R}^n$ is the derivative of X taken at the point y. Using the standard Euclidean coordinates, $\mathbf{D}X$ is the matrix with entries $\partial X^i/\partial x^j$.

Since the vector field $\psi \mapsto \mathbf{D}X(F_t(x)) \circ \psi$ on $L(\mathbb{R}^n, \mathbb{R}^n)$ (depending explicitly on t and on the parameter x) is Lipschitz in ψ, uniformly in (t, x) in a neighborhood of every (t_0, x_0), by the remark following 1.3.6 it follows that $\psi(t, x)$ is continuous in (t, x).

We claim that $\mathbf{D}F_t(x) = \psi(t, x)$. To show this, fix t and x, set $\theta(s, h) = F_s(x + h) - F_s(x)$, and write

$$\begin{aligned}
\theta(t, h) - \psi(t, x) \cdot h &= \int_0^t \{X(F_s(x+h)) - X(F_s(x))\}\, ds \\
&\quad - \int_0^t [\mathbf{D}X(F_s(x)) \circ \psi(s, x)] \cdot h\, ds \\
&= \int_0^t \mathbf{D}X(F_s(x)) \cdot [\theta(s, h) - \psi(s, x) \cdot h]\, ds \\
&\quad + \int_0^t \{X(F_s(x+h)) - X(F_s(x)) \\
&\quad - \mathbf{D}X(F_s(x)) \cdot [F_s(x+h) - F_s(x)]\}\, ds.
\end{aligned}$$

Since X is of class C^1, given $\varepsilon > 0$, there is a $\delta > 0$ such that $\|h\| < \delta$ implies the second term is dominated in norm by

$$\int_0^t \varepsilon \|F_s(x+h) - F_s(x)\|\, ds,$$

which is, in turn, smaller than $A\varepsilon\|h\|$ for a positive constant A by lemma 1.3.6. By Gronwall's inequality we obtain

$$\|\theta(t, h) - \psi(t, x) \cdot h\| \leq C\varepsilon\|h\|,$$

for a constant C. It follows that $\mathbf{D}F_t(x) \cdot h = \psi(t, x) \cdot h$. Thus both partial derivatives of $F_t(x)$ exist and are continuous; therefore $F_t(x)$ is of class C^1.

We prove $F_t(x)$ is C^r by induction on r. Begin with the equation defining F_t :

$$\frac{d}{dt} F_t(x) = X(F_t(x))$$

so

$$\frac{d}{dt} \frac{d}{dt} F_t(x) = \mathbf{D}X(F_t(x)) \cdot X(F_t(x))$$

and

$$\frac{d}{dt} \mathbf{D}F_t(x) = \mathbf{D}X(F_t(x)) \cdot \mathbf{D}F_t(x)).$$

Since the right-hand sides are C^{r-1}, so are the solutions by induction. Thus F itself is C^r. ■

Again there is an analogous result for the evolution operator $F^{\rho}_{t,t_0}(x)$ for a time-dependent vector field $X(x, t, \rho)$, which depends on extra parameters ρ in some other Euclidean space, say $\mathbb{R}^m$. If X is C^r, then $F^{\rho}_{t,t_0}(x)$ is C^r in all variables and is C^{r+1} in t and t_0.

Suspenstion Trick. The variable ρ can be easily dealt with by suspending X to a new vector field obtained by appending the trivial differential equation $\rho' = 0$; this defines a vector field on $\mathbb{R}^n \times \mathbb{R}^m$ and the basic existence and uniqueness theorem may be applied to it. The flow on $\mathbb{R}^n \times \mathbb{R}^m$ is just $F_t(x, \rho) = (F_t^\rho(x), \rho)$.

Other Approaches and Results. For another more "modern" proof of the basic existence and uniquenss theorem based directly on the implicit function theorem applied in function spaces, see Abraham, Marsden, and Ratiu [1988]. That alternative proof has a technical advantage: it works easily for other types of differentiability assumptions on X or on F_t, such as Hölder or Sobolev differentiability; this result is due to Ebin and Marsden [1970].

An interesting result called the *rectification theorem*, whose proof can be found in Arnold [1983] and Abraham, Marsden, and Ratiu [1988], shows that near a point x_0 satisfying $X(x_0) \neq 0$, the flow can be transformed by a change of variables so that the integral curves become straight lines moving with unit speed. This shows that, in effect, nothing interesting happens with flows away from equilibrium points *as long as one looks at the flow only locally and for short time.*

The Notion of a Flow. The mapping F gives a locally unique integral curve c_u for each $u \in U_0$, and for each $t \in I, F_t = F|(U_0 \times \{t\})$ maps U_0 to some other set. It is convenient to think of each point u being allowed to "flow for time t" along the integral curve c_u (see Figure 1.3.2). This is a picture of a U_0 "flowing," and the system (U_0, a, F) is a local flow of X, or ***flow box***.

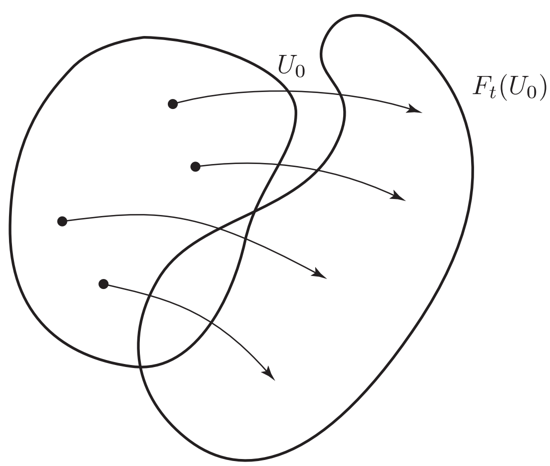

Figure 1.3.2. The flow of a vector field

Global Uniqueness of Integral Curves. While integral curves need not always exist globally, if they do, they are always unique.

Proposition 1.3.8 (**Global Uniqueness**). *Suppose c_1 and c_2 are two integral curves of X in U and that for some time t_0, $c_1(t_0) = c_2(t_0)$. Then $c_1 = c_2$ on the intersection of their domains.*

Proof. Suppose $c_1 : I_1 \to U$ and $c_2 : I_2 \to U$. Let $I = I_1 \cap I_2$, and let $K = \{t \in I \mid c_1(t) = c_2(t)\}$; K is closed since c_1 and c_2 are continuous. We will now show that K is open. From the basic existence and uniqueness result in Theorem 1.3.3, K contains some neighborhood of t_0. For $t \in K$ consider c_1^t and c_2^t, where $c^t(s) = c(t+s)$. Then c_1^t and c_2^t are integral curves satisfying $c_1(t) = c_2(t)$. By local uniqueness, they agree on some neighborhood of 0. Thus some neigborhood of t lies in K, and so K is open. Since I is connnected, $K = I$. ■

Completeness and the Lifetime of a Trajectory. Other global issues center on considering the flow of a vector field as a whole, extended as far as possible in the t-variable. In fact, by uniqueness, it makes sense to look at the largest interval in the positive and negative t-directions on which one has a solution. We make this formal as follows.

Definition 1.3.9. *Given an open set U and a vector field X on U, let $\mathcal{D}_X \subset U \times \mathbb{R}$ be the set of $(x,t) \in U \times \mathbb{R}$ such that there is an integral curve $c : I \to U$ of X with $c(0) = x$ with $t \in I$. The vector field X is* ***complete*** *if $\mathcal{D}_X = U \times \mathbb{R}$. A point $x \in U$ is called σ-**complete**, where $\sigma = +, -,$ or $\pm$, if $\mathcal{D}_X \cap (\{x\} \times \mathbb{R})$ contains all (x,t) for $t > 0, < 0,$ or $t \in \mathbb{R}$, respectively. Let $T^+(x)$ (resp. $T^-(x)$) denote the sup (resp. inf) of the times of existence of the integral curves through x; $T^+(x)$ resp. $T^-(x)$ is called the* ***positive (negative) lifetime of*** *x.*

Thus, X is complete iff each integral curve can be extended so that its domain becomes $]-\infty, \infty[$; i.e., $T^+(x) = \infty$ and $T^-(x) = -\infty$ for all $x \in U$.

Examples

A. Any linear vector field A on $\mathbb{R}^n$ is complete. Indeed, the integral curve through an initial condition $x_0 \in \mathbb{R}^n$, namely $e^{tA}x_0$ is defined for all t.

B. For $U = \mathbb{R}^2$, let X be the constant vector field $X(x,y) = (0,1)$. Then X is complete since the integral curve of X through (x,y) is $t \mapsto (x, y+t)$.

C. On $U = \mathbb{R}^2 \backslash \{0\}$, the same vector field is not complete since the integral curve of X through $(0,-1)$ cannot be extended beyond $t = 1$;

in fact as $t \to 1$ this integral curve tends to the point $(0,0)$. Thus $T^+(0,-1) = 1$, while $T^-(0,-1) = -\infty$.

D. On $\mathbb{R}$ consider the vector field $X(x) = 1 + x^2$. This is not complete since the integral curve c with $c(0) = 0$ is $c(\theta) = \tan\theta$ and thus it cannot be continuously extended beyond $-\pi/2$ and $\pi/2$; i.e., $T^\pm(0) = \pm\pi/2$. ♦

Here are some general properties of flow domains.

Proposition 1.3.10. *Let $U \subset \mathbb{R}^n$ be open and $X \in \mathfrak{X}^r(M), r \geq 1$. Then*

i *$\mathcal{D}_X \supset U \times \{0\}$;*

ii *$\mathcal{D}_X$ is open in $U \times \mathbb{R}$;*

iii *there is a unique C^r mapping $F_X : \mathcal{D}_X \to U$ such that the mapping $t \mapsto F_X(x,t)$ is an integral curve at x for all $x \in U$;*

iv *for $(x,t) \in \mathcal{D}_X, (F_X(x,t), s) \in \mathcal{D}_X$ iff $(m, t+s) \in \mathcal{D}_X$; in this case*

$$F_X(x, t+s) = F_X(F_X(x,t), s).$$

The idea of the proof is as follows. Parts **i** and **ii** follow from the local existence theory. In **iii**, we get a unique map $F_X : \mathcal{D}_X \to U$ by the global uniqueness and local existence of integral curves: $(x,t) \in \mathcal{D}_X$ when the integral curve $x(s)$ through x exists for $s \in [0,t]$. We set $F_X(x,t) = x(t)$. To show F_X is C^r, note that in a neighborhood of a fixed x_0 and for small t, it is C^r by local smoothness. To show F_X is globally C^r, first note that **iv** holds by global uniqueness. Then in a neighborhood of the compact set $\{x(s) | s \in [0,t]\}$ we can write F_X as a composition of finitely many C^r maps by taking short enough time steps so the local flows are smooth. ■

Definition 1.3.11. *Let $U \subset \mathbb{R}^n$ be open and $X \in \mathfrak{X}^r(U), r \geq 1$. Then the mapping F_X is called the* ***integral*** *of X, and the curve $t \mapsto F_X(x,t)$ is called the* ***maximal integral curve*** *of X at x. In case X is complete, F_X is called the* ***flow*** *of X.*

Thus, if X is complete with flow F, then the set $\{F_t \mid t \in \mathbb{R}\}$ is a group of diffeomorphisms on U, sometimes called a ***one-parameter group of diffeomorphisms***. Since $F_n = (F_1)^n$ (the n-th power), the notation F^t is sometimes convenient and is used where we use F_t. For incomplete flows, **iv** says that $F_t \circ F_s = F_{t+s}$ wherever it is defined. Note that $F_t(x)$ is defined for $t \in]T^-(x), T^+(x)[$. The reader should write out similar definitions for the time-dependent case and note that the lifetimes depend on the starting time t_0.

Criteria for Completeness. There are a number of conditions that are convenient for checking completeness. We begin with one of the most basic ones.

Proposition 1.3.12. *Let X be a C^r vector field on an open subset U of $\mathbb{R}^n$, where $r \geq 1$. Let $c(t)$ be a maximal integral curve of X such that for every finite open interval $]a, b[$ in the domain $]T^-(c(0)), T^+(c(0))[$ of $c, c(]a, b[)$ lies in a compact subset of U. Then c is defined for all $t \in \mathbb{R}$. If $U = \mathbb{R}^n$, this holds provided $c(t)$ lies in a bounded set.*

Proof. It suffices to show that $a \in I, b \in I$, where I is the inteval of definition of c. Let $T_n \in]a, b[, t_n \to b$. By compactness we can assume some subsequence $c(t_{n(k)})$ converges, say, to a point x in U. Since the domain of the flow is open, it contains a neighborhood of $(x, 0)$. Thus, there are $\varepsilon > 0$ and $\tau > 0$ such that integral curves starting at points (such as $c(t_{n(k)})$) for large k) closer than ε to x persist for a time longer than τ. This serves to extend c to a time greater than b, so $b \in I$ since c is maximal. Similarly, $a \in I$. ■

A direct corollary of this result relies on the notion of the support of a vector field. The ***support*** of a vector field X defined on an open set $U \subset \mathbb{R}^n$ is defined to be the closure of the set $\{x \in U | X(x) \neq 0\}$ regarded as a subset of $\mathbb{R}^n$.

Corollary 1.3.13. *A C^r vector field on an open set U with compact support contained in U is complete.*

Completeness corresponds to well-defined dynamics persisting eternally. In some circumstances (shock waves in fluids and solids, singularities in general relativity, etc.) one has to live with incompleteness or overcome it in some other way. Because of its importance we give two additional criteria. In the first result we use the notation $X[f] = \mathbf{d}f \cdot X$ for the derivative of f in the direction X. Here $f : U \to \mathbb{R}$ and $\mathbf{d}f$ stands for the derivative map. In standard coordinates on $\mathbb{R}^n$,

$$\mathbf{d}f(x) = \left(\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n}\right) \quad \text{and} \quad X[f] = \sum_{i=1}^{n} X^i \frac{\partial f}{\partial x^i}.$$

Proposition 1.3.14. *Suppose X is a C^r vector field on $\mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}$ is a C^1 **proper map**; that is, if $\{x_n\}$ is any sequence in $\mathbb{R}^n$ such that $f(x_n) \to a$, then there is a convergent subseqence $\{x_{n(i)}\}$. Suppose there are constants $K, L \geq 0$ such that*

$$|X[f](m)| \leq K|f(m)| + L \quad \text{for all} \quad m \in E.$$

Then the flow of X is complete.

Proof. From the chain rule we have $(\partial/\partial t)f(F_t(m)) = X[f](F_t(m))$, so that

$$f(F_t(m)) - f(m) = \int_0^t X[f](F_\tau(m))\,d\tau$$

Applying the hypothesis and Gronwall's inequality we see that $|f(F_t(x))|$ is bounded and hence relatively compact on any finite t-interval, so as f is proper, a repetition of the argument in the proof of 1.3.12 applies. ■

Proposition 1.3.15. *Let X be a C^r vector field on $\mathbb{R}^n$. Let σ be an integral curve of X. Assume $\|X(\sigma(t))\|$ is bounded on finite t-intervals. Then $\sigma(t)$ exists for all $t \in \mathbb{R}$.*

Proof. Suppose $\|X(\sigma(t))\| \leq A$ for $t \in]a, b[$ and let $t_n \to b$. For $t_n < t_m$ we have

$$\|\sigma(t_n) - \sigma(t_m)\| \leq \int_{t_n}^{t_m} \|\sigma'(t)\|\,dt = \int_{t_n}^{t_m} \|X(\sigma(t))\|\,dt \leq A|t_m - t_n|.$$

Hence $\sigma(t_n)$ is a Cauchy sequence and therefore, converges. Now argue as in 1.3.12. ■

Examples

A. Let X be a C^r vector field, $r \geq 1$, on the manifold U admitting a ***first integral***, i.e., a function $f : U \to \mathbb{R}$ such that $X[f] = 0$. If all level sets $f^{-1}(r), r \in \mathbb{R}$ are compact, X is complete. Indeed, each integral curve lies on a level set of f so that the result follows by Proposition 1.3.14.

B. Suppose

$$X(x) = A \cdot x + B(x),$$

where A is a linear operator of $\mathbb{R}^n$ to itself and B is ***sublinear***; i.e., $B : \mathbb{R}^n \to \mathbb{R}^n$ is C^r with $r \geq 1$ and satisfies $\|B(x)\| \leq K\|x\| + L$ for constants K and L. We shall show that X is complete. Let $x(t)$ be an integral curve of X on the bounded interval $[0, T]$. Then

$$x(t) = x(0) + \int_0^t (A \cdot x(s) + B(x(s)))\,ds$$

Hence

$$\|x(t)\| \leq \|x(0)\| \int_0^t (\|A\| + K)\|x(s)\|\,ds + Lt.$$

By Gronwall's inequality,

$$\|x(t)\| \leq (LT + \|x(0)\|)e^{(\|A\|+K)t}.$$

Hence $x(t)$ remains bounded on bounded t-intervals, so the result follows by Proposition 1.3.12.

C. We claim that the flow of the equations

$$\begin{aligned}\dot{x} &= v\\ \dot{y} &= x - x^3 - v\end{aligned}$$

is complete. To see this, note that, as we saw in equation (1.1.8) and the following discussion of dissipation, that the energy

$$E(x, v) = \frac{1}{2}v^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4.$$

is decreasing. However, because of the positive quadratic term in v and the positive quartic term in x, any sublevel set $E(x, v) \leq C$ is compact. However, any trajectory that starts in such a set stays in that set. Hence trajectories a priori stay bounded, and hence can be continued indefinitely in time.

D. Here is a more sophisticated version of the preceding example. Consider the equations for a moving particle of mass m in a potential field in $\mathbb{R}^n$, namely $\ddot{\mathbf{q}}(t) = -(1/m)\nabla V(\mathbf{q}(t))$, for $V : \mathbb{R}^n \to \mathbb{R}$ a smooth function. We shall prove that *if there are constants* $a, b \in \mathbb{R}, b \geq 0$ *such that* $(1/m)V(\mathbf{q}) \geq a - b\|\mathbf{q}\|^2$, *then every solution exists for all time*. To show this, rewrite, as usual, the second order equations as a first order system

$$\begin{aligned}\dot{\mathbf{q}} &= (1/m)\mathbf{p}\\ \dot{\mathbf{p}} &= -\nabla V(\mathbf{q})\end{aligned}$$

and note, as before that the energy

$$E(\mathbf{q}, \mathbf{p}) = \frac{1}{2m}\|\mathbf{p}\|^2 + V(\mathbf{q})$$

is a first integral—that is, is constant in time. Thus, for any solution $(\mathbf{q}(t), \mathbf{p}(t))$ we have $\beta = E(\mathbf{q}(t), \mathbf{p}(t)) = E(\mathbf{q}(0), \mathbf{p}(0)) \geq V(\mathbf{q}(0))$. We can assume $\beta > V(\mathbf{q}(0))$, i.e., $\mathbf{p}(0) \neq 0$, for if $\mathbf{p}(t) \equiv 0$, then the conclusion is trivially satisifed; thus there exists a t_0 for which $\mathbf{p}(t_0) \neq 0$ and by time translation we can assume that $t_0 = 0$. Thus we have

$$\begin{aligned}\|\mathbf{q}(t)\| \leq \|\mathbf{q}(t) - \mathbf{q}(0)\| + \|\mathbf{q}(0)\| &\leq \|\mathbf{q}(0)\| + \int_0^t \|\dot{\mathbf{q}}(s)\|\, ds\\ &= \|\mathbf{q}(0)\| + \int_0^t \sqrt{2\left[\beta - \frac{1}{m}V(\mathbf{q}(s))\right]}\, ds\\ &\leq \|\mathbf{q}(0)\| + \int_0^t \sqrt{2(\beta - a + b\|\mathbf{q}(s)\|^2)}\, ds\end{aligned}$$

or in differential form

$$\frac{d}{dt}\|\mathbf{q}(t)\| \leq \sqrt{2(\beta - a + b\|\mathbf{q}(t)\|^2)}$$

whence

$$t \leq \int_{\|\mathrm{q}(0)\|}^{\|\mathrm{q}(t)\|} \frac{du}{\sqrt{2(\beta - a + bu^2)}} \tag{1.3.3}$$

Now let $r(t)$ be the solution of the differential equation

$$\frac{d^2 r(t)}{dt^2} = -\frac{d}{dr}(a - br^2)(t) = 2br(t),$$

which, as a second order equation with constant coefficients, has solutions for all time for any initial conditions. Choose

$$r(0) = \|\mathbf{q}(0)\|, [\dot{r}(0)]^2 = 2(\beta - a + b\|\mathbf{q}(0)\|^2)$$

and let $r(t)$ be the corresponding solution. Since

$$\frac{d}{dt}\left(\frac{1}{2}\dot{r}(t)^2 + a - br(t)^2\right) = 0,$$

it follows that $(1/2)\dot{r}(t)^2 + a - br(t)2 = (1/2)\dot{r}(0)^2 + a - br(0)^2 = \beta$, i.e.,

$$\frac{dr(t)}{dt} = \sqrt{2(\beta - a + br(t)^2)}$$

whence

$$t = \int_{\|\mathrm{q}(0)\|}^{r(t)} \frac{du}{\sqrt{s(\beta - \alpha + \beta u^2}} \tag{1.3.4}$$

Comparing these two expressions (see the Exercises) and taking into account that the integrand is > 0, it follows that for any finite time interval for which $\mathbf{q}(t)$ is defined, we have $\|\mathbf{q}(t)\| \leq r(t)$, i.e., $\mathbf{q}(t)$ remains in a compact set for finite t-intervals. But then $\dot{\mathbf{q}}(t)$ also lies in a compact set since $\|\dot{\mathbf{q}}(t)\| \leq 2(\beta - a + b\|\mathbf{q}(s)\|^2)$. Thus by 1.3.12, the solution curve $(\mathbf{q}(t), \mathbf{p}(t))$ is defined for any $t \geq 0$. However, since $(\mathbf{q}(-t), \mathbf{p}(-t))$ is the value at t of the integral curve with initial conditions $(-\mathbf{q}(0), -\mathbf{p}(0))$, it follows that the solution also exists for all $t \leq 0$.

The following counterexample shows that the condition $V(\mathbf{q}) \geq a - b\|\mathbf{q}\|^2$ cannot be relaxed much further. Take $n = 1$ and $V(q) = -\varepsilon^2 q^{2+(4/\varepsilon)}/8, \varepsilon > 0$. Then the equation

$$\ddot{q} = \varepsilon(\varepsilon + 2)q^{1+(4/\varepsilon)}/4$$

has the solution $q(t) = 1/(t-1)^{\varepsilon/2}$, which cannot be extended beyond $t = 1$. ♦

The following is proved by a study of the local existence theory; we state it for completeness only.

Proposition 1.3.16. *Let X be a C^r vector field on $U, r \geq 1, x_0 \in U$, and $T^+(x_0)(T^-(x_0))$ the positive (negative) lifetime of x_0. Then for each $\varepsilon > 0$, there exists a neighborhood V of x_0 such that for all $x \in V, T^+(x) > T^+(x_0) - \varepsilon$ (respectively, $T^-(x_0) < T^-(x_0) + \varepsilon$). [One says that $T^+(x_0)$ is a lower semi-continuous function of x.]*

Corollary 1.3.17. *Let X_t be a C^r time-dependent vector field on $U, r \geq 1$, and let x_0 be an **equilibrium** of X_t, i.e., $X_t(x_0) = 0$, for all t. Then for any T there exists a neighborhood V of x_0 such that any $x \in V$ has integral curve existing for time $t \in [-T, T]$.*