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The Qualitative Analysis of a Difference Equation of Population Growth

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Summary

The difference equation $f_b: [0, 1] \to [0, 1]$ defined by $f_b(x) = b\,x(1-x)$ is studied. In particular complete qualitative information is obtained for the parameter value $b = 3.83$. For example the number of fixed points of $(f_b)^i$ is given by

$$N_i = 1 + \left(\frac{1+\sqrt{5}}{2}\right)^i + \left(\frac{1-\sqrt{5}}{2}\right)^i.$$

In mathematical models of population growth of a single species, the map $f_b: [0, 1] \to [0, 1]$ defined by $f_b(x) = b\,x(1-x)$ has played a prominent role recently. Here the parameter b satisfies $0 \le b \le 4$. See for example Hoppensteadt and Hyman [6], Li and Yorke [9], May and Oster [10].

This map f_b has a very beautiful bifurcation phenomena as a discrete (semi-) dynamical system.

Here x may be thought of as a normalized population, $f_b(x)$ the population a generation later, $f_b(f_b(x)) = (f_b)^2(x)$ the population 2 generations later, etc. Without the quadratic term, the equation is a version of the standard equation of first year calculus texts of unlimited population growth, or exponential growth. The quadratic term puts a limit which could come from crowding effects, limited food, etc.

The parameter b can be thought of as measuring "the essence of life". For $b < 1$, the population dies, i.e., goes to zero under iteration for any initial population. If b satisfies $1 < b < 3$, the population survives at the single non-trivial fixed point of the map f_b. As b increases past 3, f_b has 2 stable periodic points of period 2, then 4 of period 4, etc. When b passes the parameter value $b_3 = 3.57\ldots$, there are an infinite number of periodic points of f_b and for $b_3 < b \le 4$, the dynamics is of chaotic type. See e.g. Li and Yorke [9] for further discussion.

While there has been much numerical analysis of the dynamics in the chaotic region, as in [7], a qualitative analysis has not been carried out for any value of the parameter b, $b_3 < b \leq 4$. It is the purpose to do that here by relating the problem to the recent literature on dynamical systems.

For a background on dynamical systems see [12].

We show

Theorem: Let $b = 3.83$. Then the map $f_b: [0,1] \to [0,1]$ is structurally stable (it satisfies "Axiom A"). Furthermore the sole attractor is a periodic orbit of period 3 and the number of other periodic points N_i of f_b of period i (i.e. fixed points of $(f_b)^i$) is given exactly by the formula (the zeta function)

$$\exp \sum_{i=0}^{\infty} \frac{N_i}{i} t^i = \frac{1}{(1-t)(1-t-t^2)}$$

or

$$N_i = 1 + \left(\frac{1+\sqrt{5}}{2}\right)^i + \left(\frac{1-\sqrt{5}}{2}\right)^i$$

$$= 1 + F_{i+2} + F_{i-2}$$

where the Fibonacci numbers F_i satisfy:

$$F_0 = 0, \quad F_1 = 1,$$

$$F_i = F_{i-1} + F_{i-2}, \quad i = 2, 3, \ldots$$

For example $N_i = 2, 4, 5, 8, 12, 19, 30, 48, 77, 124$, for $i = 1, \ldots, 10$. Furthermore the complete dynamical behavior is given rather explicitly. There is a 1-point source $\{0\}$, and the only other source is a (hyperbolic) set homeomorphic to a Cantor set Ω and on Ω, the dynamics has a representation, given explicitly below, as a subshift of the shift map on 2 symbols.

One goal here is to bring into better relation, certain aspects of theoretical work on dynamical systems, numerical analysis and some equations of applied mathematics. Also it is useful to see how a complicated dynamical problem, given explicitly and simply can be analyzed asymptotically in a satisfactory way. We would like to acknowledge helpful conversations with Hyman, May, Newhouse and Yorke.

Now for the proof. We used the machine to prove a lemma.

Lemma 1: Let $f_* = (f_b)^3$ be the third iterate with $b = 3.83$. Then
$f_*(.5) = .503896$,
$f_*(.515) = .511815$,
$f_*(.9566) = .956933$,
$f_*(.9575) = .957442$,
$f_*(.155857) = .156061$,
$f_*(.159) = .157835$.

Define intervals as follows:

$$\alpha = (0, .155857), \quad \beta = (.159, .5),$$
$$\gamma = (.515, .9566), \quad \delta = (.9575, 1).$$

It can be seen easily that $f^{-1}(\alpha)$, $f^{-1}(\beta)$, $f^{-1}(\gamma)$ each have 2 components: denote them using the natural order as $f^{-1}(\alpha)=\alpha^0 \cup \alpha^1$, $f^{-1}(\beta)=\beta^0 \cup \beta^1$, $f^{-1}(\gamma)=\gamma^0 \cup \gamma^1$.

The following is easily checked.

Lemma 2:

$$\begin{aligned} &\alpha \text{ contains } \alpha^0, \beta^0,\\ &\beta \text{ contains } \gamma^0,\\ &\gamma \text{ contains } \gamma^1, \beta^1,\\ &\delta \text{ contains } \alpha^1. \end{aligned}$$

If $x \notin \alpha \cup \beta \cup \gamma \cup \delta$, $|df_b^3(x)|<1$.

Consider the induced matrix, summarizing the information in the lemma,

$$(*) \qquad \begin{array}{c} \\ \alpha \\ \beta \\ \gamma \\ \delta \end{array} \begin{array}{c} \begin{array}{cccc} \alpha & \beta & \gamma & \delta \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Note that the sole singular point of f_b is .5.

Lemma 3: The non-wandering set consists of the fixed source $\{0\}$, the attractor of period 3 of the theorem, and a subshift of finite type (a hyperbolic "repellor") given by the matrix

$$(**) \qquad \begin{array}{c} \\ \beta \\ \gamma \end{array} \begin{array}{c} \begin{array}{cc} \beta & \gamma \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \end{array}$$

Proof: First by *, δ is wandering as nothing maps into it; similarly, except for a single source $\{0\}$, all of α wanders. Then β, γ determine the matrix (**), and otherwise everything wanders to the attractor of period 3.

It remains to see that the invariant set in $\beta \cup \gamma$ is the subshift of finite type S_A given by $A=\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. This follows from a lemma of Fatou [4; pp. 72—73] which states (in so many words) that if $f: \mathbb{R} \to \mathbb{R}$ is a polynomial map and if each singular point wanders to a periodic attractor, then the rest of the non-wandering set is a repellor. See Jakobsen [8; Prop. A, p. 99] or Guckenheimer [5; p. 101].

One finds the zeta function as in [3] for example.

Concluding remarks: Since the assumption about f being a polynomial seems alien to this subject (see the proposed functions in [1], [10]), we offer two alternative ways of proceeding. First by techniques of [10] and [8], we can prove the

Lemma: If $f: [0, 1] \to [0, 1]$ is C^1 and each singular point tends toward a periodic attractor, then the remainder of tne non-wandering set is topologically semi-conjugate

$$\begin{array}{ccc} A & \longrightarrow & A \\ h \downarrow & & \downarrow h \\ S_A & \underset{\text{shift}}{\longrightarrow} & S_A \end{array}$$

to the repellor S_A, where A is determined as above. Here $N_i \geq \text{trace}\, A^i$, where the inequality may hold because the semi-conjugacy collapses some line intervals (which are just permuted by f) to points. Block [2] points out that such an h does preserve the topological entropy.

Secondly one can prove directly with a computer that such an h exists and is 1-to-1 by finding a neighborhood U of Λ and an integer k such that $\left|\frac{d(f^k)}{dx}\right| > 1$ for all x such that $x, fx, \ldots, f^{k-1}x \in U$. We verified this for f_b, $v = 3.83$ with $k = 6$ and U defined in terms of f^{-i}, $i = 1, 2, \ldots, 14$. It was found useful to make the change of variables $x \mapsto m(x) = b - 2bx$ yielding a parameterization in terms of the derivative m of f_b. One gets the smoothly conjugate map $g(m) = \frac{m^2}{2} + b - \frac{b^2}{2}$.

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