Diffusive behaviour in extended completely integrable systems

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Horward University
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Scattering shift in integrable systems

KdV:

$$\dot{u} = u''' + u\,u'$$

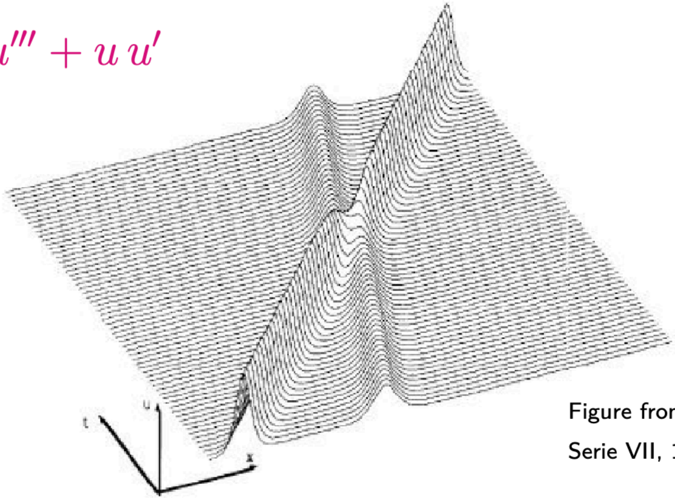
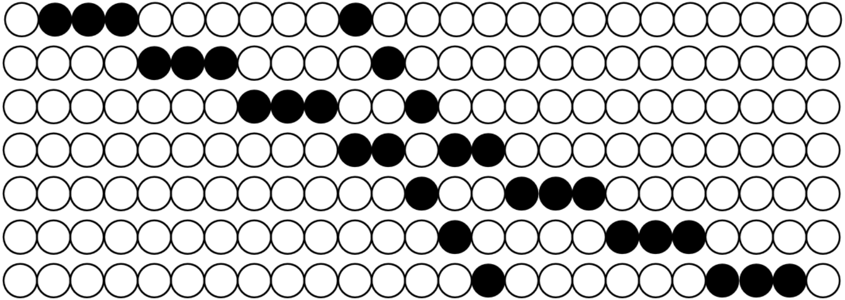


Figure from Rendiconti di Matematica, Serie VII, 11, p.351-376, 1991

Other integrable systems

- **Ball-Box cellular automata** (an ultradiscretization of KdV):



Pablo Ferrari and collaborators, Croydon-Sasada, ...

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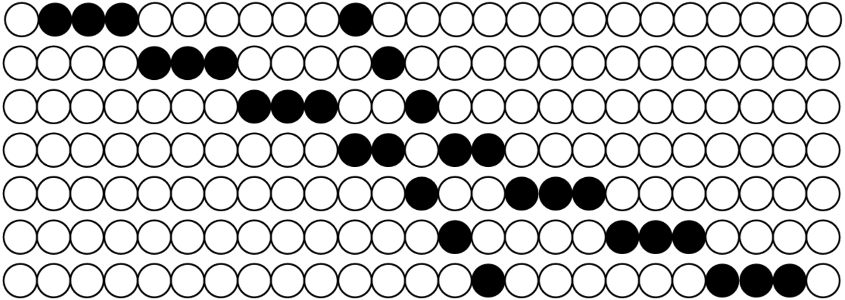

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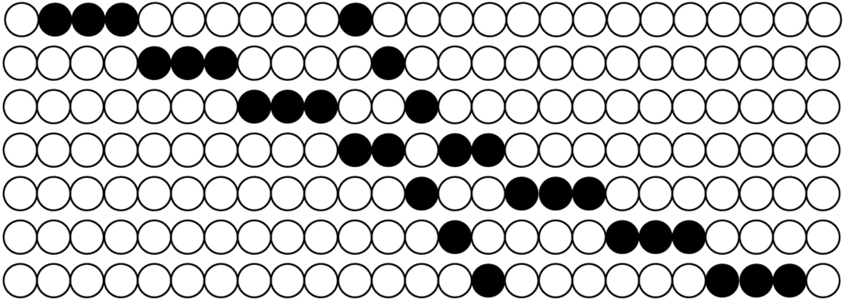

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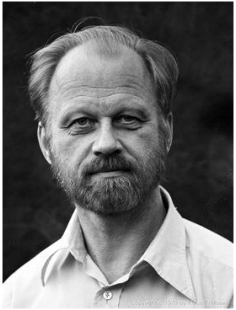
- **Toda Lattice** (H. Spohn)
- **Hard rods**

references about hard rods

- JL Lebowitz, JK Percus, J Sykes, Time evolution of the Total Distribution Function of a One-Dimensional System of Hard Rods, Physical review, 171, 1, 1968
- Boldrighini, C., Dobrushin, R.L. and Suhov, Yu.M.: Hydrodynamical limit for a degenerate model of classical statistical mechanics. Uspekhi Matem. Nauk [Russian], 35 no 4, 152 (1980)
- Boldrighini, C., Dobrushin, R.L. and Suhov, Yu.M.: One dimensional hard rod caricature of hydrodynamics. J. Stat. Phys. 31, 577–616 (1983)
- H. Spohn, Hydrodynamical Theory for Equilibrium Time Correlation Functions of Hard Rods, Annals of Physics, 141,353-364 (1982)
- C. Boldrighini, W. D. Wick, Fluctuations in a One-Dimensional Mechanical System. I. The Euler Limit, JSP, Vol. 52, Nos. 3/4, 1988
- E. Presutti, W D Wick, Macroscopic Stochastic Fluctuations in a One-Dimensional Mechanical System, Journal of Statistical Physics, Vol. 51, Nos. 5/6, 1988
- Boldrighini, C., Dobrushin, R.L. and Suhov, Yu.M.: One-Dimensional Hard-Rod Caricature of Hydrodynamics: Navier-Stokes Correction,1990
- C. Boldrighini, Yu.M. Suhov, One-Dimensional Hard-Rod Caricature of Hydrodynamics: "Navier–Stokes Correction" for Local Equilibrium Initial States, Commun. Math. Phys. 189, 577 – 590, 1997.
- B Doyon, and H Spohn, Dynamics of hard rods with initial domain wall state, J. Stat. Mech. (2017) 073210
- P.A.Ferrari, DGE Grevino, C. Franceschini, H.Spohn, Generalized hydrodynamics for size inhomogeneous hard rods, preprint 2022.

Universality in completely integrable systems



Jurgen K. Moser (1928—1999)

Every completely integrable system is (equivalent to) a collisional system

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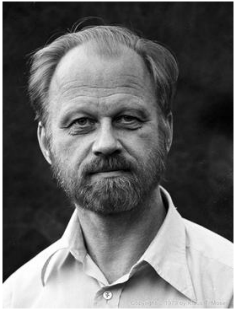

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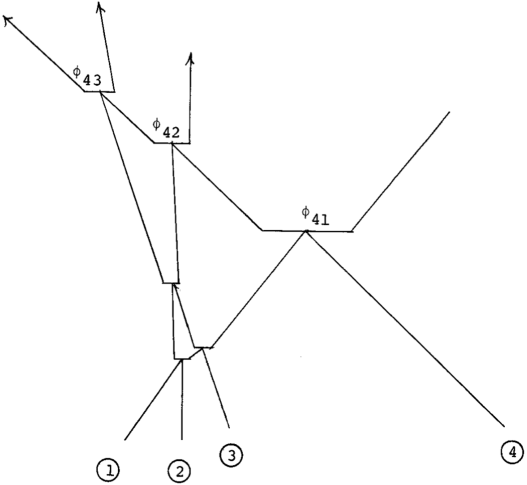
Every completely integrable system is (equivalent to) a collisional system

$\Longrightarrow$ same macroscopic behavior, where only the different scattering shifts matter.

Moser collision theorem, 1975



Marked Poisson point process

We costruct the hard rods dynamics, from a motion of non interacting points.

Let $\varepsilon << 1$, eventually $\varepsilon \to 0$

$$X^\varepsilon = \{(x, v, r)\} \qquad \text{PPP on} \quad \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$$

with intensity measure

$$\varepsilon^{-1} \rho \, dx \, d\mu(v, r), \qquad \iint d\mu(v, r) = 1, \qquad \iint (v^2 + r^2) d\mu(v, r) < +\infty.$$

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The usual hard rod case is $d\mu(v, r) = \delta_a(dr) d\tilde{\mu}(v)$.

LLN and CLT for PPP

$$\sigma = \rho \iint r d\mu(v, r), \qquad \text{mass or volume density}$$

$$\pi = \rho \iint r v d\mu(v, r), \qquad \text{momentum density.}$$

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$\varphi(x, v, r)$ test function with compact support en x.

$$\sum_{(x,v,r)\in X^\varepsilon} \varepsilon r \varphi(x,v,r) \underset{\varepsilon\to 0}{\longrightarrow} \rho \iiint r\varphi(x,v,r) dx\, d\mu(v,r) := \rho \langle\langle \varphi \rangle\rangle, \qquad a.s.$$

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$$\xi^{X,\varepsilon}(\varphi) := \varepsilon^{-1/2}\left(\varepsilon \sum_{(x,v,r)\in X^\varepsilon} r\varphi(x,v,r) - \rho\langle\langle\varphi\rangle\rangle\right) \underset{\varepsilon\to 0}{\overset{\text{law}}{\longrightarrow}} \xi^X(\varphi)$$

where ξ^X is the centered gaussian white noise with covariance

$$\mathbb{E}(\xi^X(\varphi)\xi^X(\psi)) = \rho \iiint r^2\varphi(x,v,r)\psi(x,v,r) dx\, d\mu(v,r)$$

Rods of variable length: expanded configuration

Volume occupied between a and b:

$$m_a^b(X^\varepsilon) = \begin{cases} \sum_{(x,v,r)\in X^\varepsilon} \varepsilon r 1_{[a,b)}(x) & b > a \\ -\sum_{(x,v,r)\in X^\varepsilon} \varepsilon r 1_{[b,a]}(x) & b < a. \end{cases}$$

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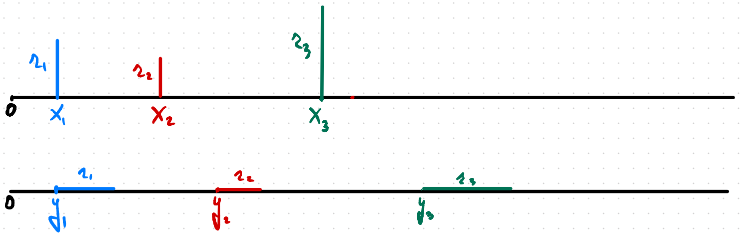
$$m_a^b(X^\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} (b-a)\sigma \qquad \text{a.s.}$$

$$X^\varepsilon \longrightarrow Y^\varepsilon = \{(y = x + m_0^x(X^\varepsilon), v, r) : (x, v, r) \in X^\varepsilon\}$$

the density of the dilated configuration is

$$\varepsilon^{-1}\bar\rho \; dy \; d\mu(v, r), \qquad \bar\rho = \frac{\rho}{1+\sigma}$$

Rods of variable length: expanded configuration



Static LLN for the hard rods

$$
\begin{aligned}
\varepsilon \sum_{(y,v,r)\in Y^\varepsilon} r\varphi(y,v,r) &= \varepsilon \sum_{(x,v,r)\in X^\varepsilon} r\varphi(x + m_0^x(X^\varepsilon), v, r) \\
&\xrightarrow[\varepsilon\to 0]{} \rho \iiint r\varphi(x + \sigma x), v, r) dx \, d\mu(v,r) \\
&= \frac{\rho}{1+\sigma} \iiint r\varphi(y,v,r) dy \, d\mu(v,r) = \bar{\rho}\langle\langle\varphi\rangle\rangle.
\end{aligned}
$$

Static CLT for the hard rods

$$\xi^{Y,\varepsilon}(\varphi) = \varepsilon^{-1/2}\left[\varepsilon \sum_{(y,v,r)\in Y^\varepsilon} r\varphi(y,v,r) - \bar{\rho}\langle\langle\varphi\rangle\rangle\right].$$

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$$\xi^{Y,\varepsilon}(\varphi) \xrightarrow[\varepsilon\to 0]{\text{law}} \xi^Y(\varphi)$$

where ξ^Y is the centered gaussian field with covariance

$$<\xi^Y(\varphi)\xi^Y(\psi)>= \bar\rho \iiint r^2 C\varphi(y,v,r)C\psi(y,v,r)dyd\mu(v,r).$$

$$C = I - \bar\rho P, \qquad P\varphi(y) = \iint r\varphi(y,v',r')\, d\mu(v',r'), \qquad \bar\rho = \frac{\rho}{1+\sigma}.$$

Dynamics (Euler Scaling)

Points $X^\varepsilon = \{(x, v, r)\}$ evolve freely without interaction:

$$x_t = x + vt$$

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Corresponding expanded (rods) configuration at time t is $\{(y_t, v, r)\}$ with

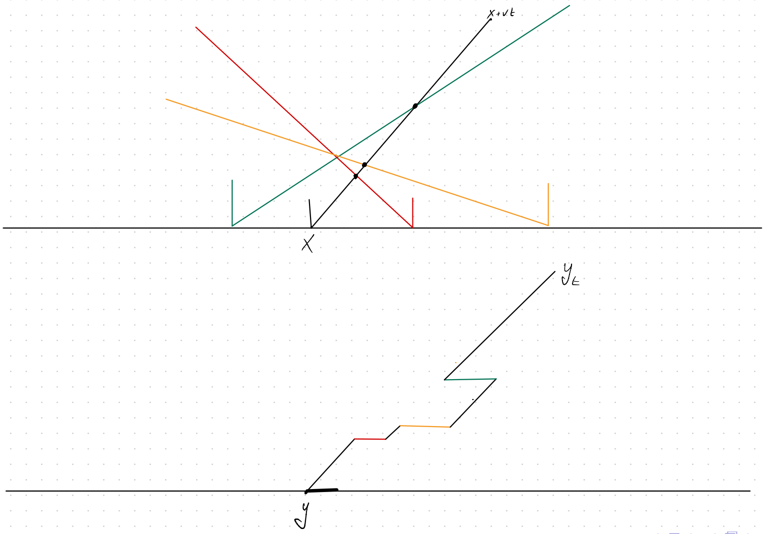
$$y_t = x + m_0^x(X^\varepsilon) + vt + j_{X^\varepsilon}(x, v, t)$$

with *collision shift* given by

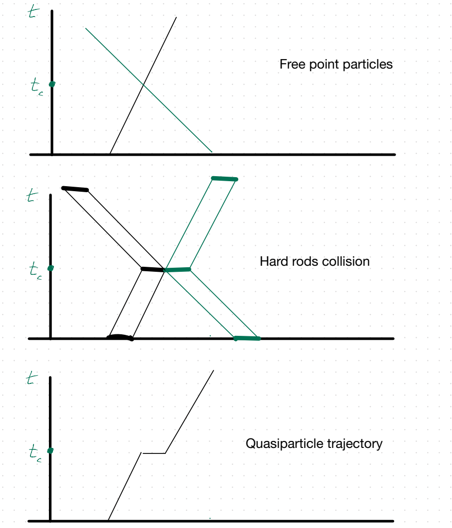
$$j_{X^\varepsilon}(x, v, t) = \varepsilon \sum_{(x',v',r')\in X^\varepsilon} r' \left(1_{[v'<v]} 1_{[x<x'<x+(v-v')t]} - 1_{[v'>v]} 1_{[x+(v-v')t<x'<x]} \right)$$

In the rods with equal length a, this is the usual elastic collision

Dynamics of *quasi-particles* or *impulses*



Dynamics of *quasi-particles* or *impulses*



Scattering shift

$$y_t = x + m_0^x(X^\varepsilon) + vt + j_{X^\varepsilon}(x, v, t)$$

with *collision shift* given by

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$$j_{X^\varepsilon}(x, v, t) \xrightarrow[\varepsilon \to 0]{a.s.} t\rho \iint r'(v - v') \, d\mu(v', r').$$

i.e. we have the effective velocity:

$$y_t \xrightarrow[\varepsilon \to 0]{a.s.} y + v^{\text{eff}}(v)t, \qquad v^{\text{eff}}(v) = v + \rho \iint r'(v - v') \, d\mu(v', r').$$

More general interaction

$$j_{X^\varepsilon}(x,v,r,t) = \varepsilon \sum_{(x',v',r')\in X^\varepsilon} \phi(v,v',r,r') \left(1_{[v'<v]}1_{[x<x'<x+(v-v')t]} - 1_{[v'>v]}1_{[x+(v-v')t<x'<x]}\right)$$

for a given *phase* function $\phi(v,v',r,r')$.

$$j_{X^\varepsilon}(x,v,t) \xrightarrow[\varepsilon\to 0]{a.s.} t\rho \iint \phi(v,v',r,r')(v-v')d\mu(v',r').$$

Fluctuation field (Euler Scaling)

Evolution of the fluctuating field for the points X^ε is trivial:

$$\xi_t^{X,\varepsilon}(\varphi) = \varepsilon^{-1/2}\left[\varepsilon \sum_{(x,v,r)\in X^\varepsilon} r\varphi(x+vt,v,r) - \rho\langle\langle\varphi\rangle\rangle\right] \xrightarrow[\varepsilon\to 0]{} \xi^X(\varphi_t)$$

where $\varphi_t(x,v,r) = \varphi(x+tv,v,r)$.

Fluctuation field (Euler Scaling)

For the rods the fluctuation field in Euler scaling is

$$\xi_t^{Y,\varepsilon}(\varphi) = \varepsilon^{-1/2}\left[\varepsilon \sum_{(y,v,r)\in Y^\varepsilon} r\varphi(y_t, v, r) - \bar{\rho}\langle\langle\varphi\rangle\rangle\right]$$

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as $\varepsilon \to 0$ this converges in law to

$$\xi_t^Y(\varphi) = \xi_0^Y(\tilde{\varphi}_t), \qquad \tilde{\varphi}_t(y, v, r) = \varphi(y + v^{\text{eff}}(v)t, v, r).$$

i.e. formally is the simple linear transport equation

$$\partial_t \xi_t^Y(y, v, r) + v^{\text{eff}}(v)\partial_y \xi_t^Y(y, v, r) = 0$$

For the usual deterministic lenght HR, this was proven in Boldrighini-Wick, JSP, 1988.

Euler Scale Hydrodynamics

$$\partial_t \xi_t^Y(y,v,r) + v^{\text{eff}}(v)\partial_y \xi_t^Y(y,v,r) = 0$$
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This is the linearization of the *Hydrodynamic Equation* out of equilibrium:

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Boldrighini, C., Dobrushin, R.L. and Suhov, Yu.M.: Hydrodynamical limit for a degenerate model of classical statistical mechanics. Uspekhi Matem. Nauk [Russian], 35 no 4, 152 (1980), JSP (1983).

Diffusive Scaling

Recall that y_t is the position at time t of the hard rod (y, v, r).

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$$y_{\varepsilon^{-1}t} - v^{\text{eff}}(v)\varepsilon^{-1}t \xrightarrow[\varepsilon\to 0]{\text{in law}} y + \sqrt{\mathcal{D}(v)}W_t(y, v)$$

where $W_t(y, v)$ is a Wiener process in t, parametrized by (y, v).

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It turns out that $W_t(y, v)$ does not depend on y, $W_t(y, v) = W_t(v)$. this is equivalent as proving that

$$j_{X^{\varepsilon}}(x, v, \varepsilon^{-1}t) - (v^{\text{eff}}(v) - v)\varepsilon^{-1}t \xrightarrow[\varepsilon\to 0]{\text{in law}} \sqrt{\mathcal{D}(v)}W_t(v). \tag{1}$$

$$\mathcal{D}(v) = \rho \iint r^2|v - \bar{v}|d\mu(\bar{v}, r).$$

Diffusion

$$\mathcal{D}(v) = \rho \iint r^2 |v - \bar{v}| d\mu(\bar{v}, r).$$

In the fixed size case this is the same diffusivity as in:

Boldrighini, C., Dobrushin, R.L. and Suhov, Yu.M.: One-Dimensional Hard-Rod Caricature of Hydrodynamics: Navier-Stokes Correction,1990

C. Boldrighini, Yu.M. Suhov, One-Dimensional Hard-Rod Caricature of Hydrodynamics: "Navier–Stokes Correction" for Local Equilibrium Initial States, Commun. Math. Phys. 189, 577 – 590, 1997.

or computed by Green-Kubo formula in

H. Spohn, Hydrodynamical Theory for Equilibrium Time Correlation Functions of Hard Rods, Annals of Physics, 141,353-364 (1982)

Full Correlation of tagged particles with the same velocity

insert picture

$$\lim_{\varepsilon\to 0} \mathrm{Cov}\left(j_{X^\varepsilon}(x, v, \varepsilon^{-1} t), j_{X^\varepsilon}(\bar{x}, v, \varepsilon^{-1} t)\right) = \lim_{\varepsilon\to 0} \mathrm{Var}\left(j_{X^\varepsilon}(x, v, \varepsilon^{-1} t)\right) = t\mathcal{D}(v)$$

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In general for different velocities they remain correlated

$$\lim_{\varepsilon\to 0} \text{Cov}\left(j_{X^\varepsilon}(x,v,\varepsilon^{-1}t), j_{X^\varepsilon}(\bar{x},\bar{v},\varepsilon^{-1}t)\right) = t\Gamma(v;\bar{v})$$

$$\Gamma(v;\bar{v}) = \frac{1}{2}\left(\mathcal{D}(v) + \mathcal{D}(\bar{v}) - |v-\bar{v}|\,\rho \iint r^2 d\mu(\bar{v},r)\right)$$

Rods fluctuation field (diffusive scaling)

$$\Xi_t^{Y,\varepsilon}(\varphi) = \varepsilon^{-1/2}\left[\varepsilon \sum_{(y,v,r)\in Y^\varepsilon} r\varphi\left[y_{\varepsilon^{-1}t} - v^{\text{eff}}(v)\varepsilon^{-1}t, v, r\right] - \frac{1}{1+\sigma}\langle\langle\varphi\rangle\rangle\right].$$

Theorem

$$\Xi_t^{Y,\varepsilon}(\varphi) \xrightarrow[\varepsilon\to 0]{law} \Xi_t^{Y}(\varphi) = \xi^Y\left(\varphi(\cdot + \sqrt{\mathcal{D}}W_t)\right).$$

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Theorem

$$\Xi_t^{Y,\varepsilon}(\varphi) \xrightarrow[\varepsilon\to 0]{law} \Xi_t^Y(\varphi) = \xi^Y\left(\varphi(\cdot + \sqrt{\mathcal{D}}W_t)\right).$$

More explicitely

$$\Xi_t^Y(\varphi) = \iiint r\varphi\left(y + \sqrt{\mathcal{D}(v)}W_t(v), v, r\right) d\xi_0^Y(y,v,r)$$

Multiplicative one dimensional noise

By Ito's formula, this is the solution of

$$d\Xi_t^Y(\varphi) = \frac{1}{2}\Xi_t^Y(\mathcal{D}(\cdot)\partial_y^2\varphi)dt - \iiint \sqrt{\mathcal{D}(v)}(\partial_y\varphi)(y,v,r)dW_t(v)d\Xi_t^Y(y,v,r)$$

or in the time integrated form:

$$\Xi_t^Y(\varphi) = \Xi_0^Y(\varphi) + \int_0^t \frac{1}{2}\Xi_s^Y(\mathcal{D}(\cdot)\partial_y^2\varphi)ds - \int_0^t \Xi_s^Y\left(\sqrt{\mathcal{D}(\cdot)}\,\partial_y\varphi\, dW_s\right),$$

where the last term is a martingale with quadratic variation

$$\int_0^t \left(\iiint \sqrt{\mathcal{D}(v)}(\partial_y\varphi)(y,v,r)d\Xi_s^Y(y,v,r)\right)^2 ds = \int_0^t \Xi_s^Y\left(\sqrt{\mathcal{D}(\cdot)}(\partial_y\varphi)\right)^2 ds.$$

with expectation given by

$$t\bar{\rho}\iiint \mathcal{D}(v)r^2(C\partial_y\varphi)^2(y,v,r)dy\, d\mu(v,r).$$

comparison with chaotic dynamics

formally this is an equation with *multiplicative noise*:

$$\partial_t \Xi_t^Y(y,v,r) = \frac{1}{2}\mathcal{D}(v)\partial_y^2 \Xi_t^Y(y,v,r) + \partial_y \Xi_t^Y(y,v,r)\dot{W}_t(v)$$

$$\mathbb{E}\left(\dot{W}_t(v)\dot{W}_s(v')\right) = \delta(t-s)\Gamma(v,v')$$

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In a **chaotic dynamics** we expect instead an *additive space-time white noise* (∞-dimensional O-U process):

$$\partial_t \Xi_t^Y(y,v,r) = \frac{1}{2}\mathcal{D}(v)\partial_y^2 \Xi_t^Y(y,v,r) + \sqrt{\mathcal{D}(v)}\,\partial_y \dot{W}_{t,y}(v)$$

$$\mathbb{E}\left(\dot{W}_{t,y}(v)\dot{W}_{s,y'}(v)\right) = \delta(t-s)\delta(y-y')$$

Conserved quantities after diffusive scaling

Choose $\varphi_{k,\bar v}(x,v,r) = e^{i2\pi xk}\varphi(r)\delta(v-\bar v)$

$$\hat\varphi(k,\bar v,t) := \Xi_t^{Y,\varepsilon}(\varphi_{k,\bar v}) = \int d\xi^Y(y,r,v)e^{i2\pi k(y+\sqrt{D(\bar v)}W_t)}\varphi(r)\delta(v-\bar v)$$
$$= \xi^Y\left(e^{i2\pi k(\cdot+\sqrt{D(\bar v)}W_t)}\varphi(\cdot)\delta(\cdot-\bar v)\right)$$

satisfy the SDE

$$d\hat\varphi(k,\bar v,t) = -\frac{(2\pi k)^2}{2}\mathcal{D}(\bar v)\hat\varphi(k,\bar v,t) + i2\pi k\sqrt{\mathcal{D}(\bar v)}\hat\varphi(k,\bar v,t)dW_t(\bar v).$$

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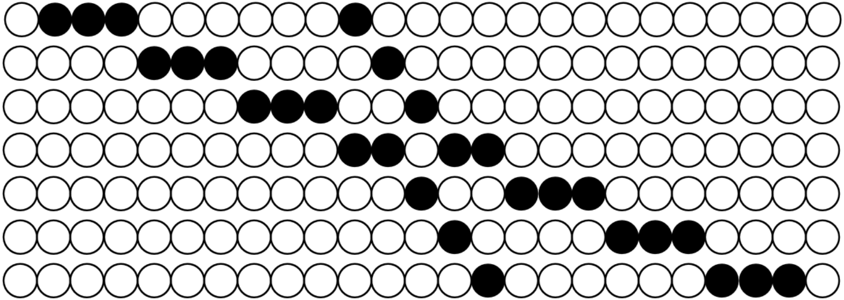
$|\hat{\varphi}(k,\bar{v},t)|^2 = |\hat{\varphi}(k,\bar{v},0)|^2$ for any k,
a persistence on the diffusive macroscopic scale of the complete integrability of the dynamics.

Extension to Box-Ball cellular automata in low density limits

(with Hayate Suda, Makiko Sasada)



Ball-Box in low density limits

By a theorem of Gabrielli and Ferrari, there exists stationary and translation invariant measures where the densities of k-solitons are ρ_k and solitons are *independently distributed*.
We set such measure ν_ε such that

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$$\varepsilon X_1\left(\left[\frac{t}{\varepsilon}\right]\right) - \varepsilon X_1(0) - \frac{1-\rho_2}{1+\rho_2} t \quad \xrightarrow[\varepsilon \to 0]{\nu_\varepsilon - \text{prob}} \quad 0$$

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Box-Ball diffusive scaling

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- Consequently the fluctuations of the density of 2-solitons evolve in the diffusive time scale translated rigidly by a brownian motion, like for the hard rods case.