Wall-crossing structures in Donaldson-Thomas theory

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Outline

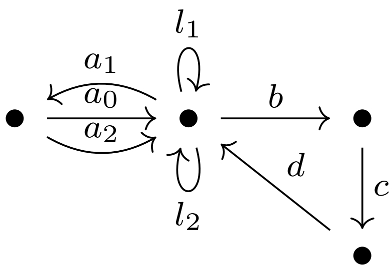
Motivation

Wall-crossing structures (a.k.a. scattering diagrams) encode counting problems in relevant categories.

- Donaldson-Thomas Theory: count semistable objects in 3-dimensional Calabi-Yau categories,
- Mirror Symmetry: count pseudo-holomorphic discs, and does not depend on the stability condition on the relevant Fukaya category.

Quiver with potential

An example: a quiver Q



with potential $W = a_1 l_1^2 l_2^3 a_0 + l_1 dcb$.

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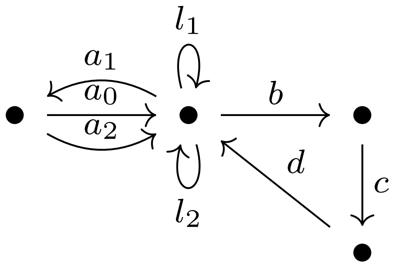

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The potential is cyclic invariant, e.g., $W = a_0 a_1 l_1^2 l_2^3 + dcbl_1$.

Quiver with potential

Notations

I = the set of vertices,
Ω = the set of arrows,
a_{ij} = the number of arrows from i to j for $i, j \in I$.

Path algebra and Jacobi algebra

Fix a field $\mathbf{k}$, we have the following

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Definition

The *Jacobi algebra* of (Q, W) is $J(Q, W) = \mathbf{k}Q/\langle \frac{\partial W}{\partial a}, a \in \Omega \rangle$.

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We let $\chi_Q(\gamma_1, \gamma_2) = -\sum_{i \in I} \gamma_1^i \gamma_2^i + \sum_{a: i \to j} \gamma_1^i \gamma_2^j$, and $\langle \gamma_1, \gamma_2 \rangle = \chi_Q(\gamma_1, \gamma_2) - \chi_Q(\gamma_2, \gamma_1)$.

Spaces of quiver representations

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$\mathbf{M}_\gamma$: the space of Q-representations of dimension $\gamma = (\gamma^i)_{i\in I}$,
$\mathbf{G}_\gamma := \prod_{i\in I} \mathrm{GL}(\gamma^i, \mathbb{C})$ acts on $\mathbf{M}_\gamma$ by conjugation,
$\mathcal{W}_\gamma := \mathrm{Tr}\, W_\gamma$: a $\mathbf{G}_\gamma$-invariant function on $\mathbf{M}_\gamma$.
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Consider the space of quiver representations up to the group action?

- Framework of stacks
- Stability conditions (can produce nice geometric objects, e.g., smooth, projective)

Stability

Stability conditions: general work by Bridgeland, Kontsevich - Soibelman, etc.

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Fix a central charge (a.k.a. stability function) $Z : \mathbb{Z}^I \to \mathbb{H}_+ := \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$, an object E (a Q rep. or $J(Q,W)$-mod.) is called *semistable* if any subobject $F \subset E$ satisfies

$$\mathrm{Arg}(F) := \mathrm{Arg}(Z(\mathrm{dim} F)) \leqslant \mathrm{Arg}(E) := \mathrm{Arg}(Z(\mathrm{dim} E)),$$

where $\mathrm{Arg}(z)$ is the argument of a complex number z.

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where $\mathrm{Arg}(z)$ is the argument of a complex number z.
Moreover, an object E is *stable* if any proper subobject F satisfies $\mathrm{Arg}(F) < \mathrm{Arg}(E)$.

King's stability

A special case:

Definition

Given a function $\zeta : \mathbb{Z}^I \to \mathbb{R}$, an object E is *semistable* if $\zeta(\dim E) = 0$, and $\zeta(\dim F) \leqslant 0$ for any $F \subset E$.

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Note that a King's stability function gives rise to a central charge $Z = -\zeta + i$.

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Define

$$\Phi_\gamma = exp(-\sum_{k\geqslant 1} \frac{e_{k\gamma}}{k^2}).$$

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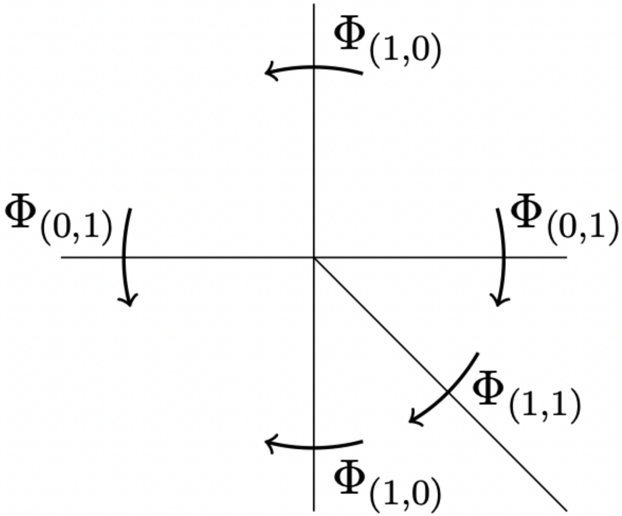
Semistable representations:

- $\zeta = (0, \zeta_2)$: $S_1^{\oplus n}$, where $S_1 : \mathbb{C} \xrightarrow{0} 0$ is stable;
- $\zeta = (\zeta_1, 0)$: $S_2^{\oplus n}$, where $S_2 : 0 \xrightarrow{0} \mathbb{C}$ is stable;
- $\zeta = (\zeta_1, -\zeta_1), \zeta_1 > 0$: $P_1^{\oplus n}$, where $P_1 : \mathbb{C} \xrightarrow{\sim} \mathbb{C}$ is stable.

Example: A_2

Scattering diagram of A_2 quiver



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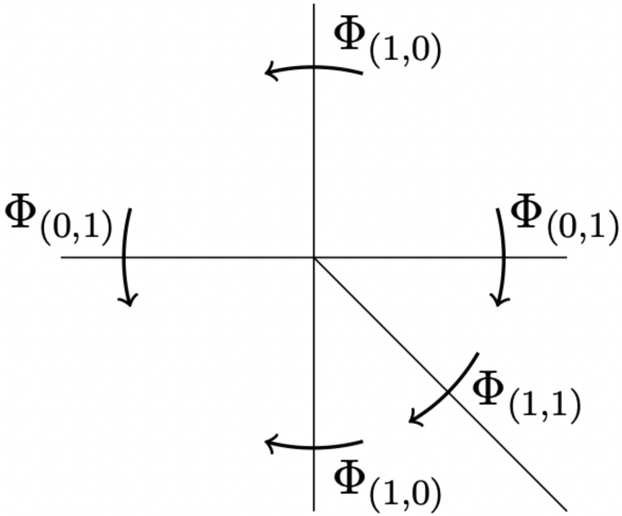

Wall-crossing formular:

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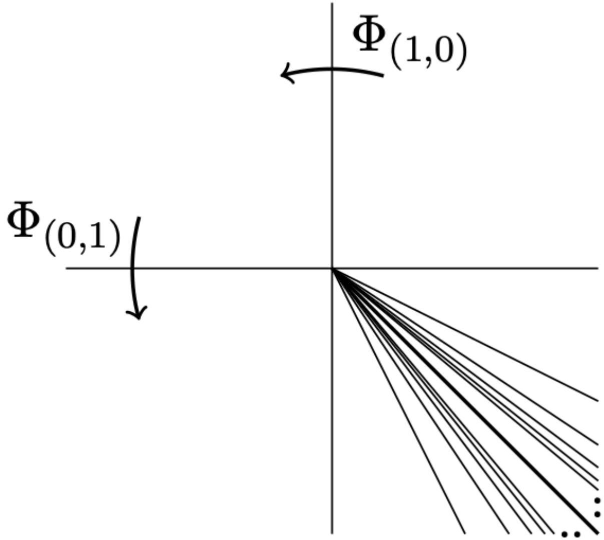

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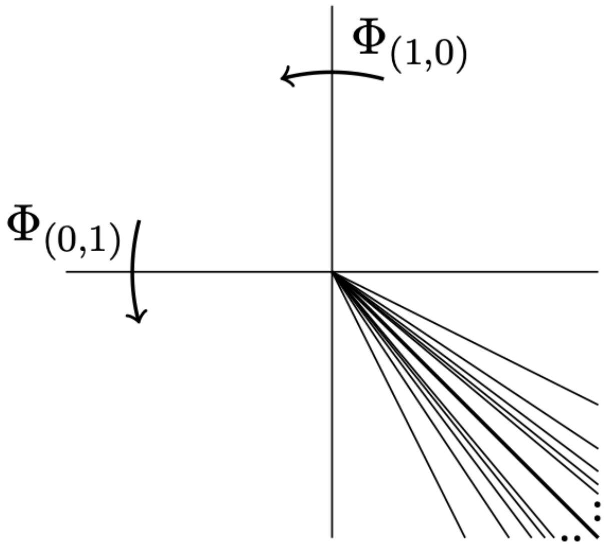

$$\Phi_{(0,1)} \cdot \Phi_{(1,0)} = \Phi_{(1,0)} \cdot \Phi_{(2,1)} \cdot \Phi_{(3,2)} \cdot \cdots \cdot \Phi_{(1,1)}^{-2} \cdot \cdots \cdot \Phi_{(2,3)} \cdot \Phi_{(1,2)} \cdot \Phi_{(0,1)}$$

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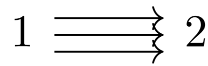

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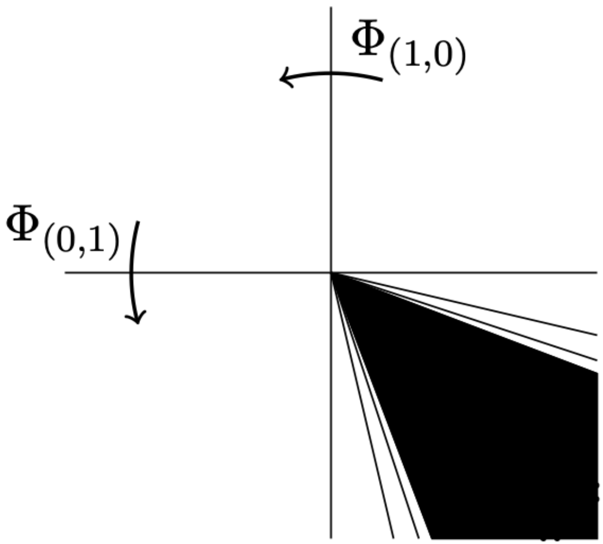

Scattering diagrams

Γ: finitely generated free abelian group, $\Gamma^+ \subset \Gamma$: a cone,

$\mathfrak{g} = \bigoplus_{\gamma \in \Gamma^+} \mathfrak{g}_\gamma$: a graded Lie algebra,

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A *scattering diagram* $\mathfrak{D}$ consists of

- a collection of codimension one closed subsets $\mathfrak{d} \in \Gamma^\vee_\mathbb{R} = Hom(\Gamma, \mathbb{R})$ knows as walls, and each wall is a convex cone in the hyperplane $\gamma^\perp \subset \Gamma^\vee_\mathbb{R}$ defined by some primitive vector $\gamma \in \Gamma^+$,

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A scattering diagram is *consistent* if for any sufficiently general path $p : [0, 1] \to \Gamma^\vee_\mathbb{R}$, the ordered product $\Phi_p = \prod \Phi^\pm_{\mathfrak{d}_i} \in \hat{G}$ given by the sequence of walls $\mathfrak{d}_i$ crossed by p depends only on the endpoints of p.

Scattering diagrams

Main approaches:

cohomological Hall algebras (COHA): Kontsevich, Soibelman, Davison, Schiffmann, Vasserot, Sala, Yang, Zhao, R., etc.

motivic Hall algebras: Kontsevich, Soibelman, Joyce, Song, Bousseau, etc.

Example: $\mathbb{P}^2$

Let $Coh(\mathbb{P}^2)$ be the abelian category of coherent sheaves on $\mathbb{P}^2$. Given $E \in Coh(\mathbb{P}^2)$, we denote $\gamma(E) = (r(E), d(E), \chi(E)) \in \mathbb{Z}^3$.

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Consider the following family of central charges:

Definition

For any $(s,t) \in U = \{(s,t) \in \mathbb{R}^2 \mid y > -x^2/2\}$, let

$$\begin{aligned} Z^{(s,t)} : \quad & \mathbb{Z}^3 \to \mathbb{C}, \\ & \gamma = (r, d, \chi) \mapsto tr + sd + r + \tfrac{3}{2}d - \chi + i(d - sr)\sqrt{s^2 + 2t}. \end{aligned}$$

Example: $\mathbb{P}^2$

Scattering diagram of coherent sheaves on $\mathbb{P}^2$