The Mathematics of Doodling

HOWARD UNIVERSITY

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Mr. Barry Long, my 7th + 8th grade
math, english, and homeroom teacher
1938 - 2022

How I would doodle...

(How do you doodle?)

Drawing circles around things

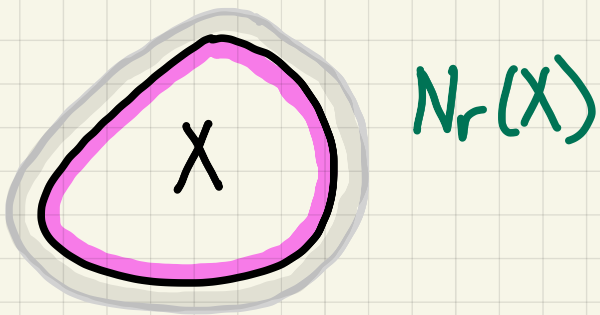
Age: preschool

We wonder: Are the doodles getting more and more circular? Why?

Prior question: What does this even mean? What precisely (mathematically) are we even doing?

Given a set X in the plane, define the

r^{th} neighborhood:



$$N_r(X) := \{ y : |y-x| \leq r \text{ for some } x \in X \}$$

Reworded question: In some sense does

$$N_r(\cdots N_r(N_r(X)) \cdots)$$

become "more and more circular"?

This suggests another question:

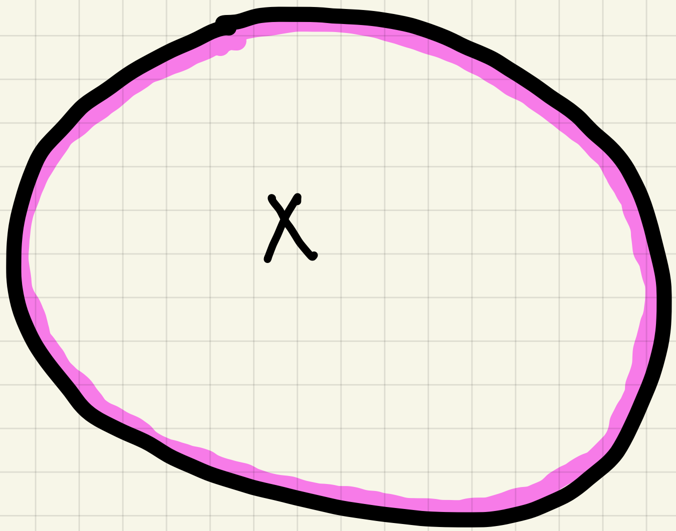
$\boxed{\text{How does the choice of } r \text{ affect the situation?}}$

How does $N_1(X)$ compare to $N_2(X)$?

Vote:

$\boxed{N_1(N_1(X)) \text{ is at least as big}}$

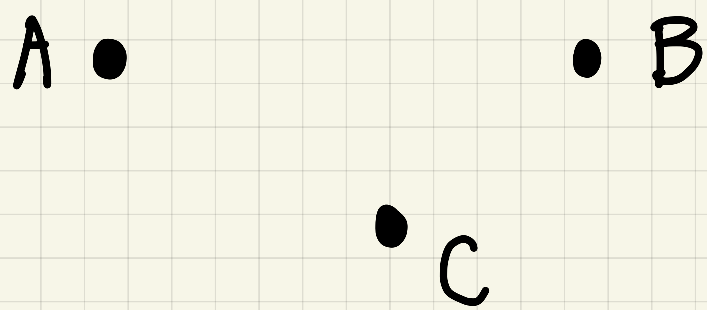
$\boxed{N_2(X) \text{ is at least as big}}$



We are inevitably led to discover

The Triangle Inequality

The shortest distance between two points is a straight line.

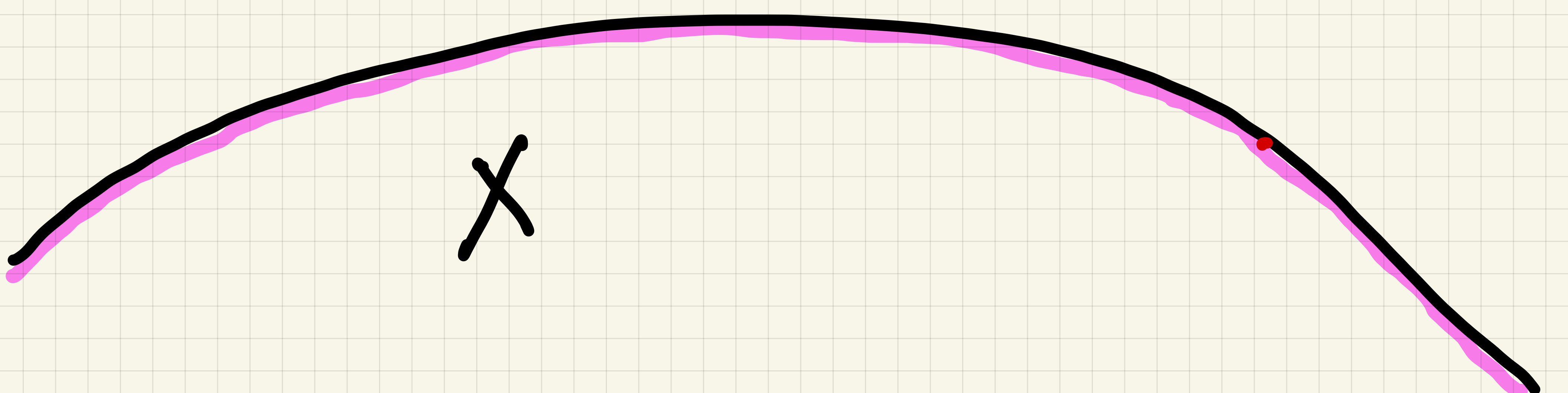


Mathematically: $|AC| + |CB| \geq |AB|$

Bringing this back to our question...

$N_a(N_b(X))$ is at least as big

$N_{a+b}(X)$ is at least as big

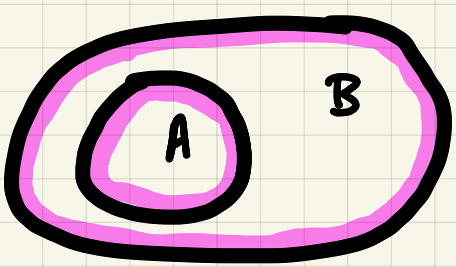


So for example $\underbrace{N_1(N_1(\cdots N_1(X)\cdots))}_{1{,}000{,}000 \text{ times}} = N_{1{,}000{,}000}(X)$

<u>Reworded question</u>: As $r \to \infty$, does $N_r(X)$ get "more circular"?

<u>Answer</u>: yes!

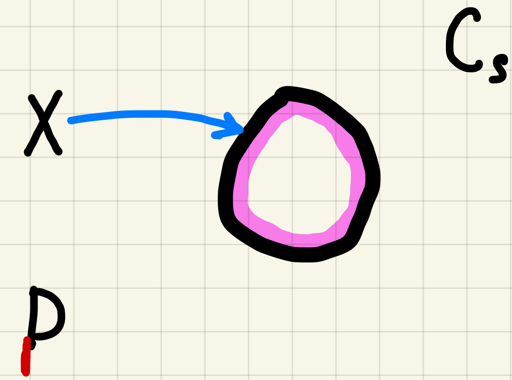
<u>Key idea</u>: If A is contained in B, then $N_r(A)$ is contained in $N_r(B)$.



(We'll use this principle again.)

Suppose p is a point in X. Let C_t be the circle of radius t centered at p. Choose an s so that C_s contains X.

So $\{p\} \subset X \subset C_s$



Now doodle!

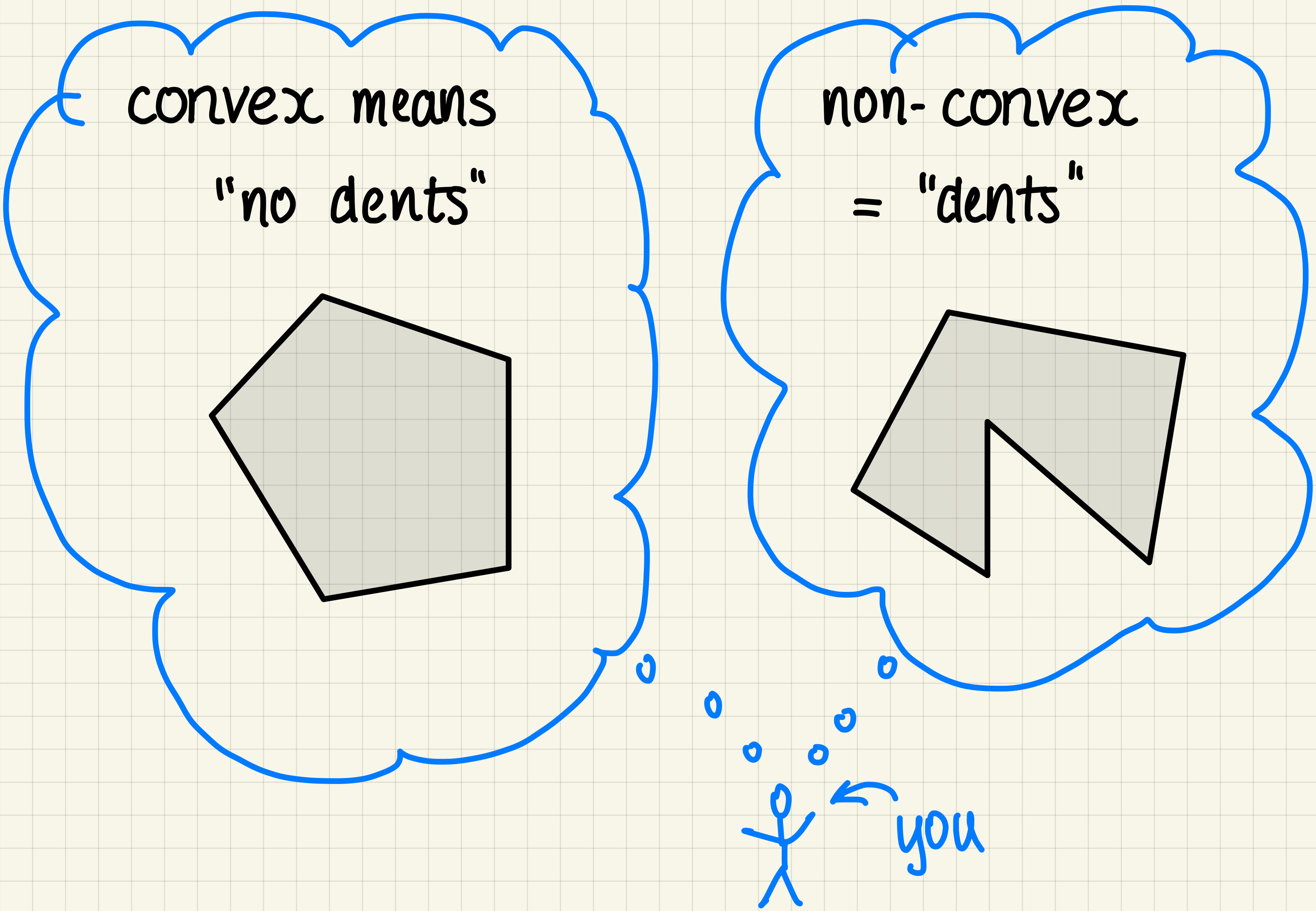
Picture when $r = 1{,}000{,}000\, s$:

Thus as $r \to \infty$, the border of $N_r(X)$ is "squeezed" between two circles, and the ratios of radii goes to 1. We declare victory!

Age : school

Let's keep going!

For simplicity let X be a convex polygon.



Age : school

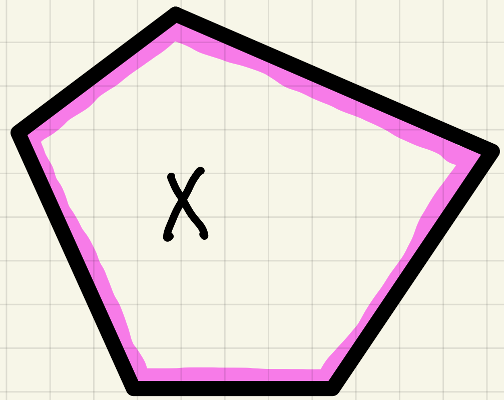
Let's keep going!

For simplicity let X be a convex polygon.

Let P denote perimeter.

Given $P(X) = 4$. What is $P(N_1(X))$?

$P(N_r(X))$



Let P denote perimeter.

Given $P(X)$. What is $P(N_r(X))$?

$P(N_r(X)) =$

Let A denote Area. Given $A(X)$ as well, what is $A(N_r(X))$?

$$\boxed{\begin{aligned} A(N_r(X)) &= A + Pr + \pi r^2 \\ P(N_r(X)) &= P + 2\pi r \end{aligned}}$$

Age: college

Side comment: $P = A'$

We can't help but notice:

$$P(N_r(X)) = \frac{d}{dr} A(N_r(X)). \quad \text{Why ?!}$$

Notice that we are inevitably led to certain insights. We have no choice.

Subtle observation:

$$A(N_r(X)) = A \cdot 1 + P \cdot r + \pi \cdot r^2$$

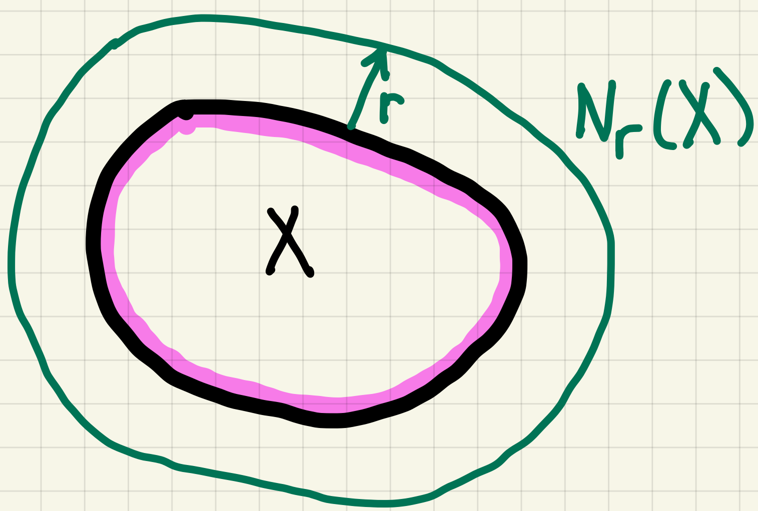
is a polynomial in r, whose coefficients have important geometric meaning.

Generalize!

Suppose X is convex, but not a polygon.

Redefinition (not fully necessary*)

We define N_r using a length r "normal vector".



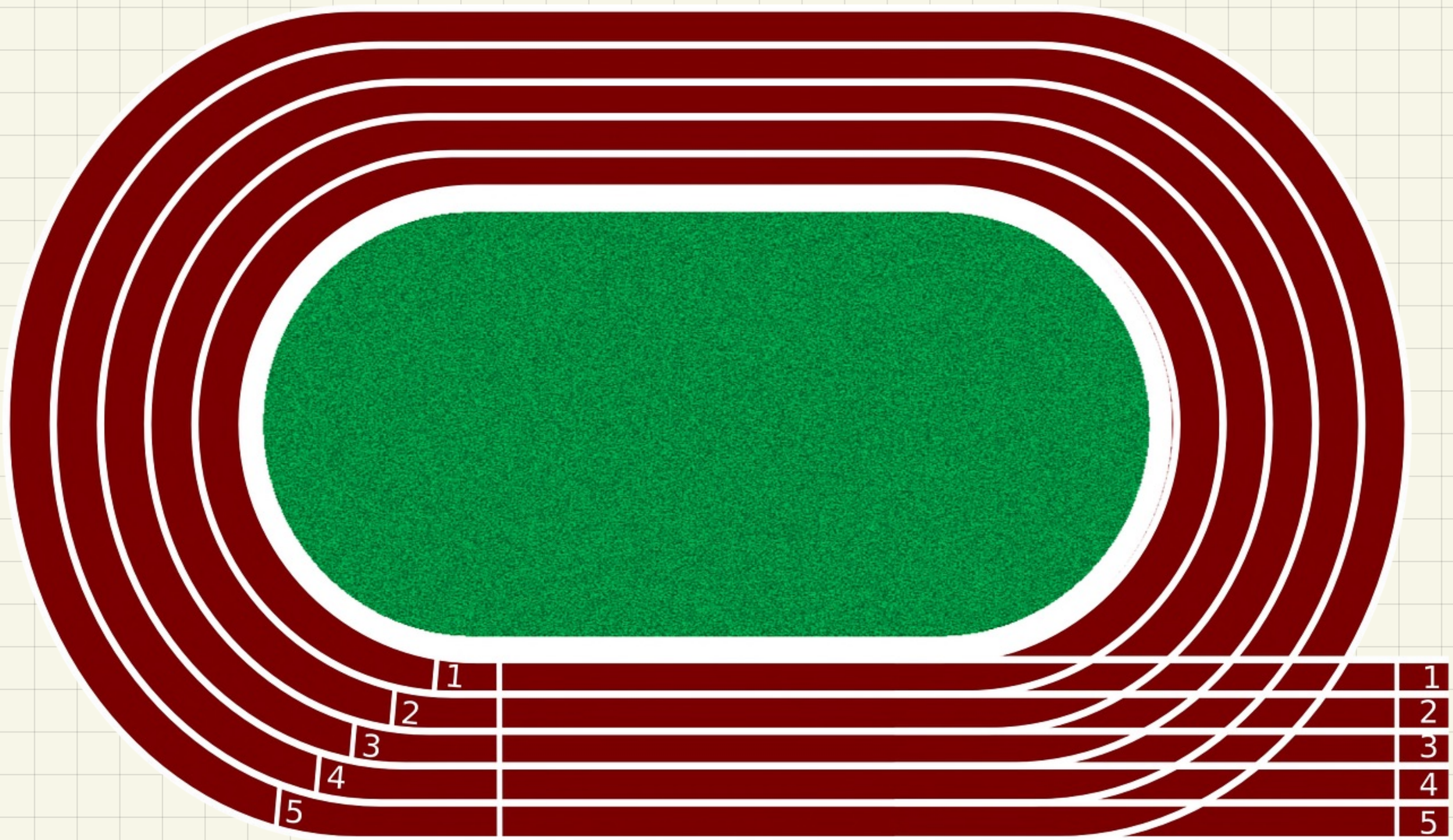
Fact The same formulas hold!

$$P(N_r(X)) = P(X) + 2\pi r$$

$$A(N_r(X)) = A(X) + P(X)r + \pi r^2$$

Why?

* so why am I doing it?


1
2
3
4
5
1
2
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5

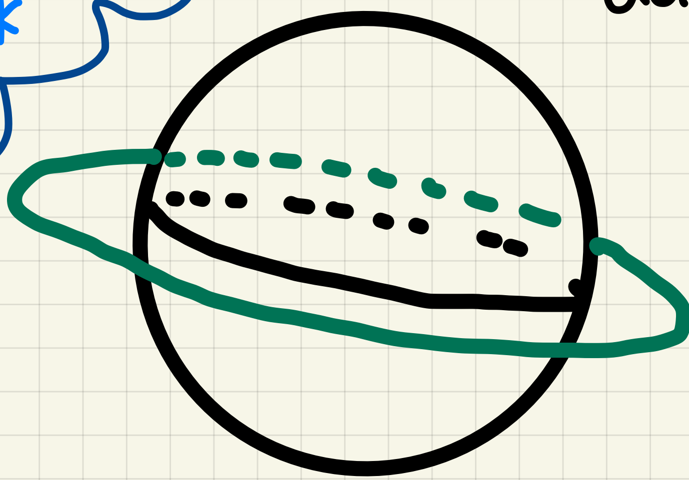
The "string-around-the-earth" puzzle

Age: school

For the purpose of this puzzle, the earth is a perfect sphere.*

String is wrapped tightly around the equator of the earth. Someone cuts the string, adds 1 m to its length, and raises the string to a height r above the ground.

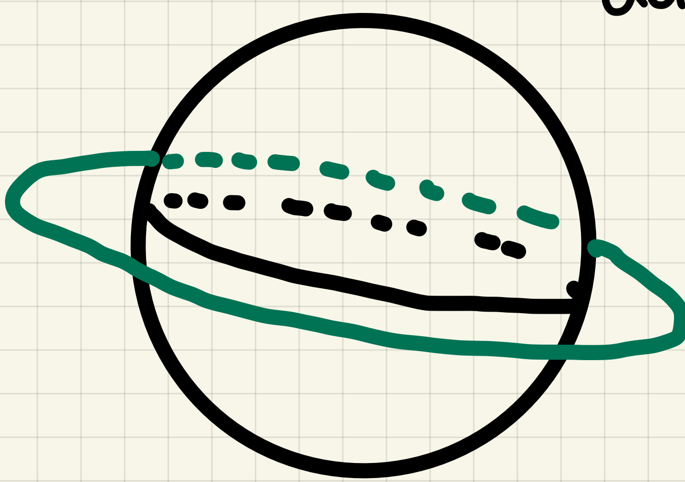
What is r?



* Interesting fact that pure mathematicians may not know: the earth is <u>not</u> a perfect sphere. More generally: The earth is far from perfect.

The "string-around-the-earth" puzzle

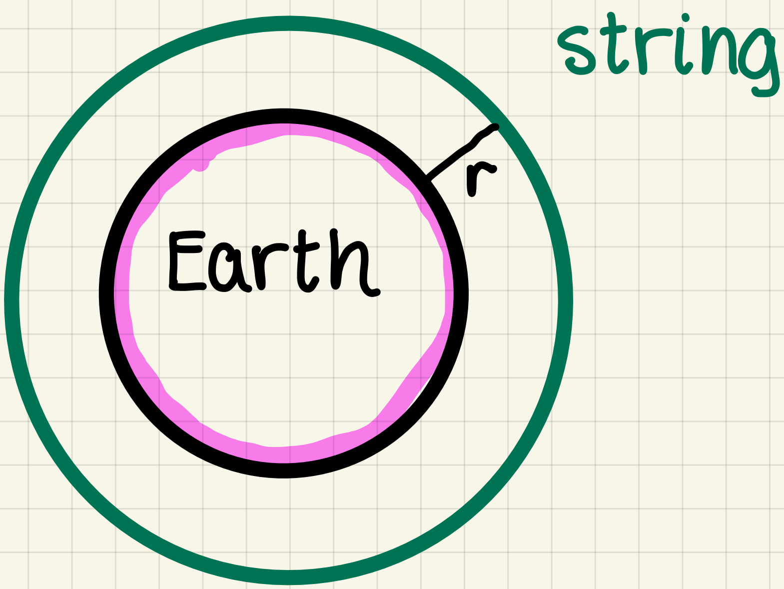
String is wrapped tightly around the equator of the earth. Someone cuts the string, adds 1 m to its length, and raises the string to a height r above the ground. What is r?



Your opinion?

10^{-9} m ?	10^{-6} m ?	10^{-3} m ?

Solution (from this ridiculously fancy point of view)



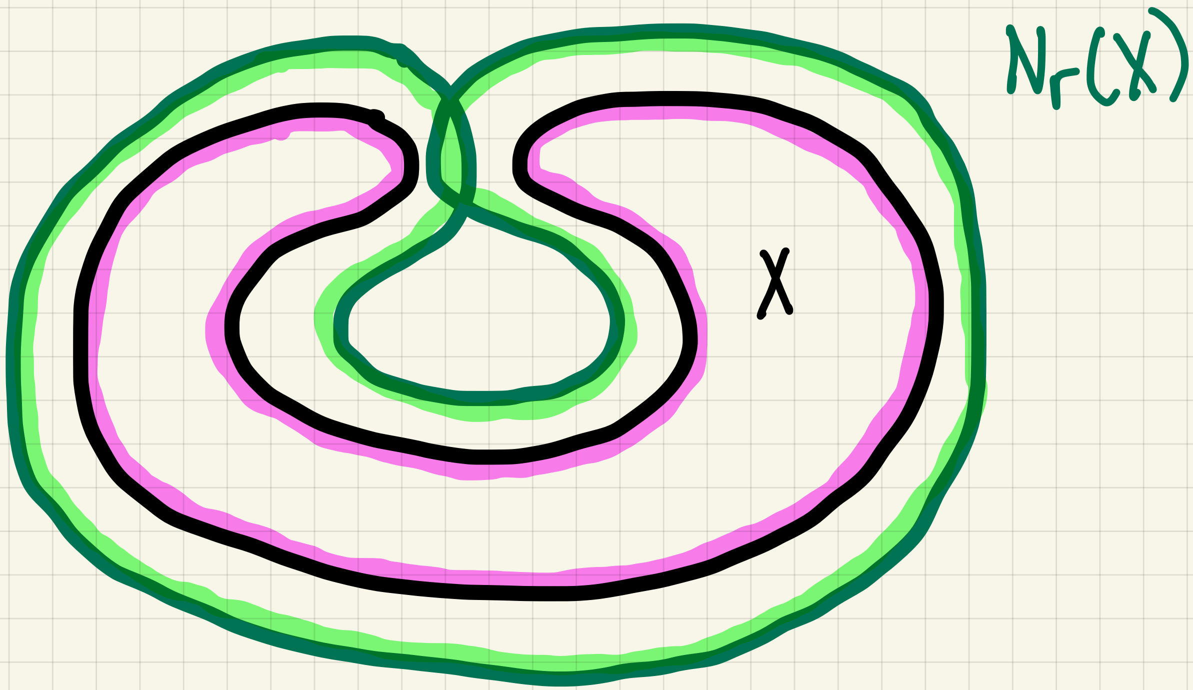
Bonus:
Mars! Asteroids!

$$P(\text{string}) = P(\text{earth}) + 2\pi r$$

$2\pi r = 1\,m$ so $r = \frac{1}{2\pi}\,m$

Generalize!

Give up convexity

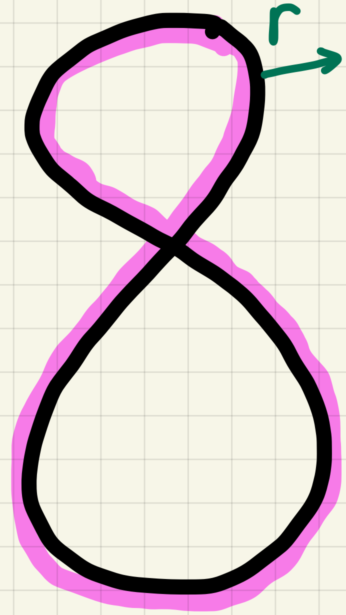


$$P(N_r(X)) = P(X) + 2\pi r$$

Is $A(N_r(X)) = A(X) + P(X)r + \pi r^2$?

Discovery!

What about a figure 8 ?



$$P(N_r(X)) = P(X) + \cancel{2\pi r}$$

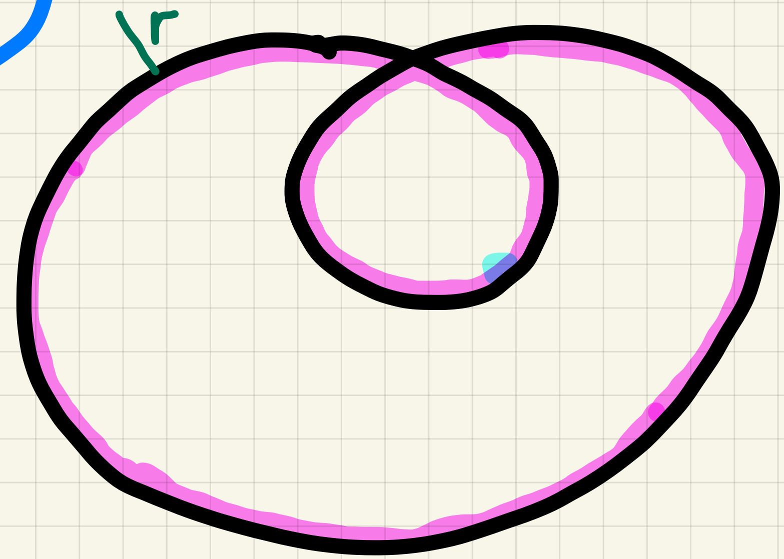
$$A(N_r(X)) \overset{?}{=} A(X) + P(X)r + \cancel{\pi r^2}$$

What is the "area" of the 8 ?

We have lost the $2\pi r$! Where did it go?

To answer this question, we experiment further.

A more complicated shape:



??!

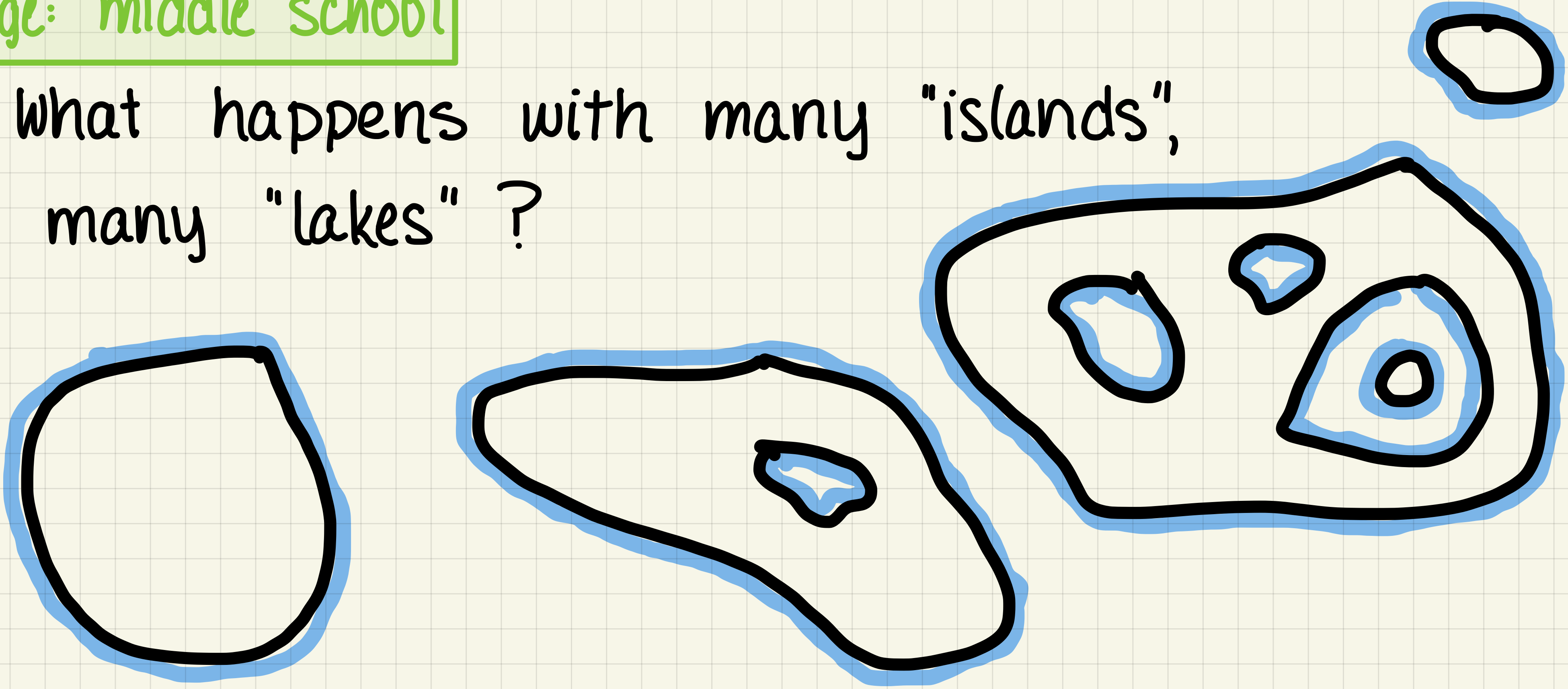
$$P(N_r(X)) = P(X) + 4\pi r$$

$$A(N_r(X)) = A(X) + P(X)r + 2\pi r^2$$

We discover the "winding number" of a loop!

Age: middle school

What happens with many "islands", many "lakes"?



<u>Answer:</u>

Age: graduate school

$$A(N_r(X)) = A(X) + P(X)r + \chi(X)r^2$$

$$P(N_r(X)) = P(X) + \chi(X)\,2\pi r$$

greek letter chi

We have discovered the euler characteristic in topology!

More generalizations:

What happens in three dimensions? or...

Let's calculate this for a box of height h, length ℓ, and width w.

Let V denote volume and A denote surface area.

$$V(N_r(X)) = V + Ar + \frac{4}{3}\pi r^3$$

$$\boxed{V(N_r(X)) = V + Ar + (l+w+h)\pi r^2 + \frac{4}{3}\pi r^3}$$

This must work for all convex bodies, right?

Let's try it out!

sphere radius R: $V(N_r(X)) =$

$$\frac{4}{3}\pi(R+r)^3 = \left(\frac{4}{3}\pi R^3\right) + \left(4\pi R^2\right)r + (4\pi R)r^2 + \frac{4}{3}\pi r^3$$

polyhedron

$$V(N_r(X)) = V(X) + A(X)r + ???\,\pi r^2 + \frac{4}{3}\pi r^3$$

A beautiful Russian problem

A Russian train company has a rule: you are not allowed luggage (boxes) whose sum of dimensions (length + width + height) exceeds 1 m. Can you "cheat" by taking an illegal box, and packing it in a legal box?

NO! Why?

boxes

Suppose X is inside Y

Then $N_r(X)$ is inside $N_r(Y)$.

So $V(N_r(X)) \leq V(N_r(Y))$

$$V(X) + A(X)r + (l_x + w_x + h_x)\pi r^2 + \frac{4}{3}\pi r^3 \leq$$

$$V(Y) + A(Y)r + (l_Y + w_Y + h_Y)\pi r^2 + \frac{4}{3}\pi r^3$$

A Hilbert problem

Can you dissect a cube and reassemble the pieces to form a regular tetrahedron?

(In dimension 2, the answer is yes: you can cut up any polygon, and rearrange the pieces to form any other polygon with the same area.)

Answer: NO. There is a new dissection invariant built from our new one-dimensional doodling invariant.

In dimension n:

$$V(N_r(X)) = V(X) + A(X)\,r + ??\,r^2 + \cdots + ??\,r^{n-1} + \begin{pmatrix}\text{volume of a} \\ \text{unit n-sphere}\end{pmatrix} r^n$$

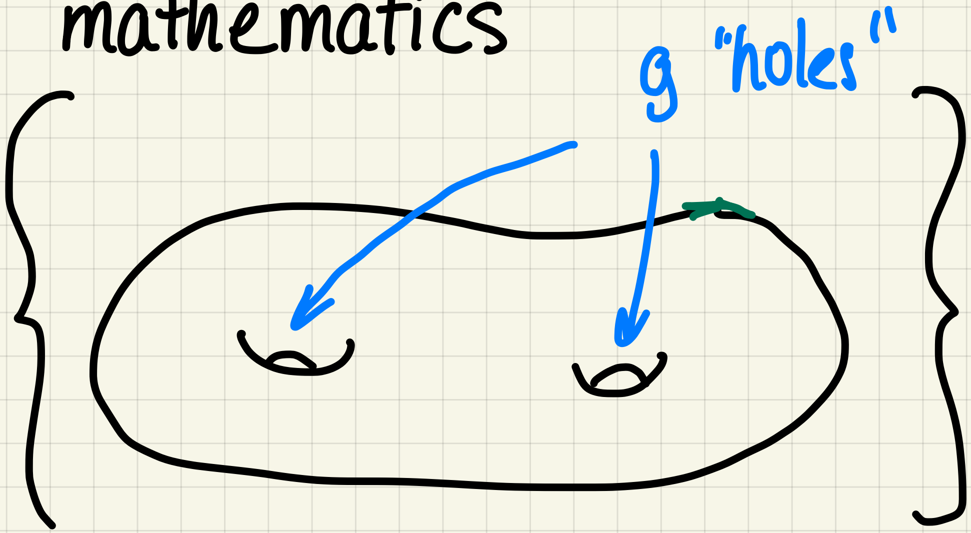
These higher invariants must be important!

Riddle:

The volume of an n-sphere of radius r is $\dfrac{\pi^{n/2}}{(n/2)!}\, r^n$

$n=0$	1	✓
$n=1$	$2r$	?!
$n=2$	πr^2	✓
$n=3$	$\frac{4}{3}\pi r^3$	?!

Links to "fancier" mathematics

Riemann: 

dimension of this "space"?

definition?

Mumford

Deligne - Mumford

size? shape (topology)?

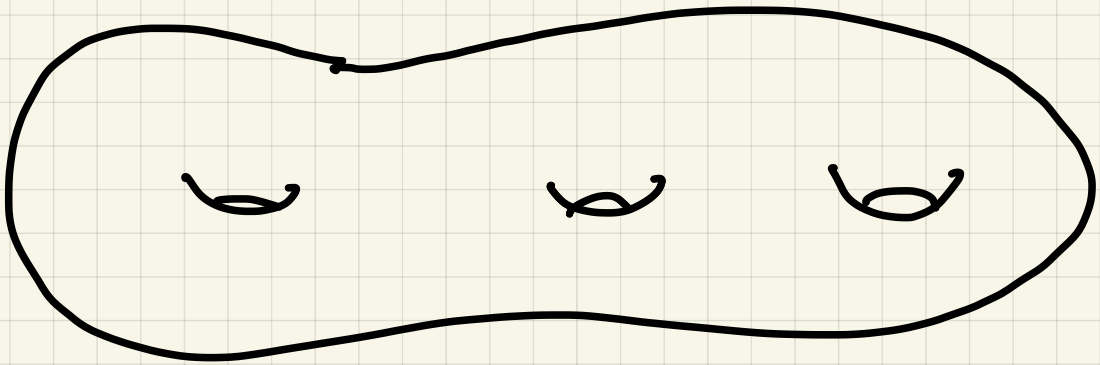
Witten

Kontsevich

Okounkov-Pandharipande

McMullen

Maryam Mirzakhani



"size" of this space

is a magic polynomial

which can be found by dissecting shape

big doodles: $r \longrightarrow \infty$

Witten's conjecture

With the right spirit of curiosity, adventure, and fearlessness, we find lurking behind even simple-looking doodles, mathematics of surprising beauty and power. Thank you!

Know what you want,
And don't get distracted.

- Maryam Mirzakhani
1977 - 2017