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WHICH ROOT DOES THE BISECTION ALGORITHM FIND?*

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A student of elementary probability may be amused by an application of probability to numerical analysis.

Let f be a continuous function on the closed interval $[a, b]$ such that $f(a)f(b) < 0$. Then f has at least one root α on (a, b) . The well-known bisection algorithm [1, p. 28] generates a sequence $\{[a_k, b_k]\}$ of intervals on which a root is known to lie. Let $a_0 = a, b_0 = b$, and define $c_k = (a_k + b_k)/2$. If $f(c_k) = 0, c_k = \alpha$, and the algorithm terminates. If $f(a_k)f(c_k) < 0, \alpha \in (a_k, c_k)$, so let $[a_{k+1}, b_{k+1}] = [a_k, c_k]$. For the other possible case, if $f(a_k)f(c_k) > 0, \alpha \in (c_k, b_k)$, so let $[a_{k+1}, b_{k+1}] = [c_k, b_k]$. $|b_k - a_k| = 2^{-k}|b_0 - a_0|$, so that the bisection algorithm is guaranteed to converge to some root of f on $[a, b]$.

If f has more than one root on $[a, b]$, a problem in [1, p. 35] asks which root the bisection algorithm usually locates. If f has n distinct, simple roots $x_1 < x_2 < \dots < x_n$ on $[a, b]$ ($f(x_i) = 0$ and $f'(x_i) \neq 0$), then it is well-known that the bisection algorithm finds the even numbered roots with probability zero. This paper shows that the probability of finding the odd numbered roots is uniform.

Let C_n denote the class of continuous functions which satisfy $f(a)f(b) < 0$ with exactly n distinct, simple roots on (a, b) . Let the roots of $f \in C_n$ be denoted by $x_1 < x_2 < \dots < x_n$. We assume that the locations of the roots are independent and distributed according to a uniform random distribution on $[a, b]$. Let $x_0 = a$ and $x_{n+1} = b$. Let $P_{i,n}$ denote the probability that the bisection algorithm converges to the i th root of f , given that $f \in C_n$. Let $Q_{i,n}$ denote the probability that

$$x_i < c_0 = (a_0 + b_0)/2 < x_{i+1},$$

for $i = 0, 1, \dots, n$, given that $f \in C_n$.

We first note that n is odd and that $P_{i,n} = 0$ for all even i since at each step, the bisection algorithm discards the subinterval of length $(b_k - a_k)/2$ which contains an even number of roots. Hence for even i, x_i will be found if and only if $c_k = (a_k + b_k)/2 = x_i$ at some step of the algorithm.

THEOREM. For n odd,

$$P_{i,n} = \begin{cases} \frac{2}{n+1} & \text{for } i \text{ odd,} \\ 0 & \text{for } i \text{ even.} \end{cases}$$

If i is even, $P_{i,n} = 0$ was shown above. The proof for i odd is by induction on n . $P_{1,1} = 1$, for if f has only one root on $[a, b]$, the bisection algorithm is guaranteed to find it.

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Assume that

$$P_{i,m} = \begin{cases} \frac{2}{m+1} & \text{for } i \text{ odd,} \\ 0 & \text{for } i \text{ even,} \end{cases}$$

for all odd $m < n$.

$Q_{i,n} = \binom{n}{i}/2^n$ since it is equal to the probability that i of n trials (roots) land on the left half of $[a, b]$.

If $x_j < c_0 = (a_0 + b_0)/2 < x_{j+1}$ for some fixed even j , then the probability of finding x_i is given by

$$\begin{cases} P_{i-j,n-j} & \text{for } i > j, \\ 0 & \text{for } i \leq j, \end{cases}$$

since the bisection algorithm will proceed on the interval $[a_1, b_1] = [c_0, b_0]$, which contains $n - j$ roots of f .

Similarly, if $x_j < c_0 < x_{j+1}$ for some fixed odd j , then the probability of finding x_i is given by

$$\begin{cases} 0 & \text{for } i > j, \\ P_{i,j} & \text{for } i \leq j. \end{cases}$$

Then

$$\begin{aligned} P_{i,n} &= \sum_{j=0}^n P(\text{finding } x_i \text{ given } x_j < c_0 < x_{j+1}) \\ &= \sum_{\substack{j=0 \\ j \text{ even}}}^{i-1} Q_{j,n} P_{i-j,n-j} + \sum_{\substack{j=i \\ j \text{ odd}}}^n Q_{j,n} P_{i,j} \\ &= Q_{0,n} P_{i,n} + Q_{n,n} P_{i,n} + \sum_{\substack{j=2 \\ j \text{ even}}}^{i-1} Q_{j,n} P_{i-j,n-j} \\ &\quad + \sum_{\substack{j=i \\ j \text{ odd}}}^{n-2} Q_{j,n} P_{i,j}, \end{aligned}$$

where we assume that $\sum_2^0 = \sum_n^{n-2} = 0$. Hence,

$$\begin{aligned} P_{i,n} &= \frac{1}{1 - Q_{0,n} - Q_{n,n}} \left[\sum_{j=2}^{i-1} Q_{j,n} P_{i-j,n-j} + \sum_{j=i}^{n-2} Q_{j,n} P_{i,j} \right] \\ &= \frac{1}{2^n - 2} \left[\sum_{j=2}^{i-1} \binom{n}{j} \frac{2}{n-j+1} + \sum_{j=i}^{n-2} \binom{n}{j} \frac{2}{j+1} \right] \\ &= \frac{1}{2^n - 2} \left[\sum_{j=2}^{n-1} \binom{n+1}{j} \right] \frac{2}{n+1} \\ &= \frac{2}{n+1} \frac{1}{2^n - 2} \sum_{j=2}^{n-1} \left[\binom{n}{j-1} + \binom{n}{j} \right] = \frac{2}{n+1}. \end{aligned}$$

We have shown that if a function f has n distinct, simple roots, the bisection algorithm is equally likely to find each odd root.

REFERENCE

S. D. CONTE AND CARL DE BOOR, *Elementary Numerical Analysis: An Algorithmic Approach*, 2nd ed., McGraw-Hill, New York, 1965.