Topology of planar sections of the skew polyhedron $\{4,6|4\}$

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Lehigh 2015, in honor of Don Davis

The general problem is due to S.P. Novikov,
who extracted it from Physics literature
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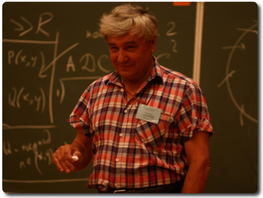
of Morse theory to multivalued functions.



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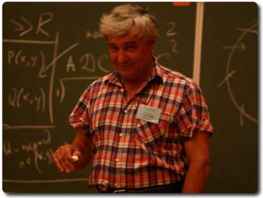

This is partly joint work

with his pupil I.A. Dynnikov.

The infinite skew polyhedron $\{4,6|4\}$ (μ-cube)

The μ-cube [1] is the universal cover of $\boldsymbol{\mu C} = \{f_{\boldsymbol{\mu C}} = \frac{1}{2}\} \subset \mathbb{T}^3$,

$$f_{\boldsymbol{\mu C}}(x,y,z) = \text{mid}\{|2x-1|,|2y-1|,|2z-1|\} : \mathbb{T}^3 \to \mathbb{R}$$

and $\mathbb{T}^3 = [0,1]^3/\simeq$.

[1] See J.H. Conway, H. Burgiel and C. Goodman-Strauss, "The Symmetries of Things", A K Peters

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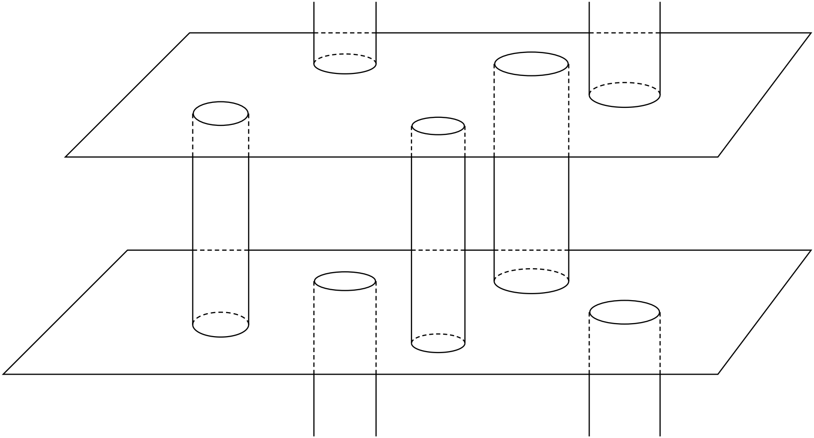
- *How do the leaves of $B = 0$ wind on $\boldsymbol{\mu C}$?*
- *How does the topology of leaves depends on the direction of B?*

A simple (but actually generic) example

Consider the genus–6 surface below, embedded in $\mathbb{T}^3$...



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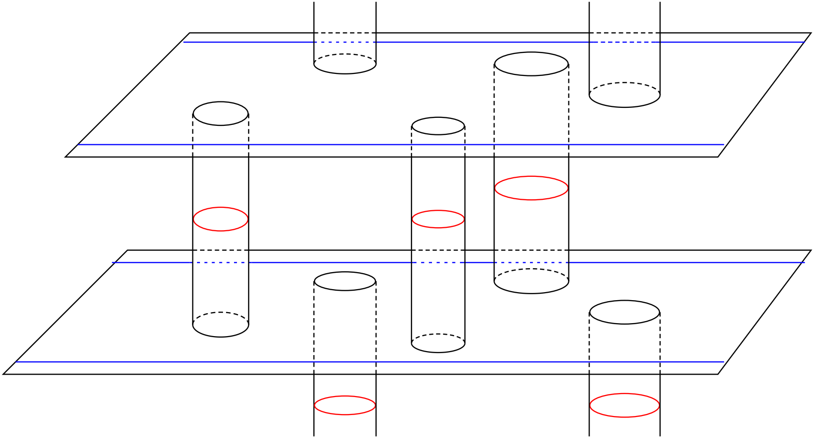
Consider the genus–6 surface below, embedded in $\mathbb{T}^3$...
and cut it with a bundle of almost horizontal planes.



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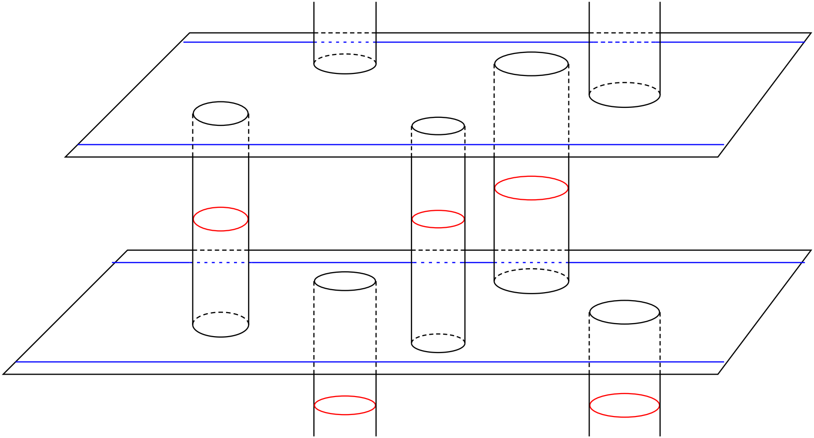
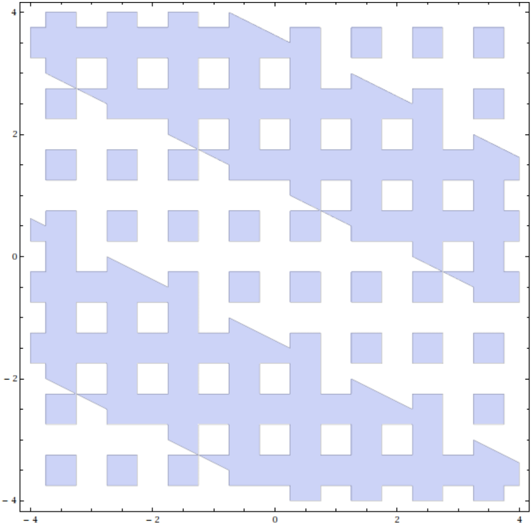
Under small perturbations, the cilinders of closed leaves will not disappear and the open leaves will be bound to fill genus-1 rank-2 surfaces.



Examples of planar sections



Planar section of the μ-cube orthogonal to $(1, 2, 10)$

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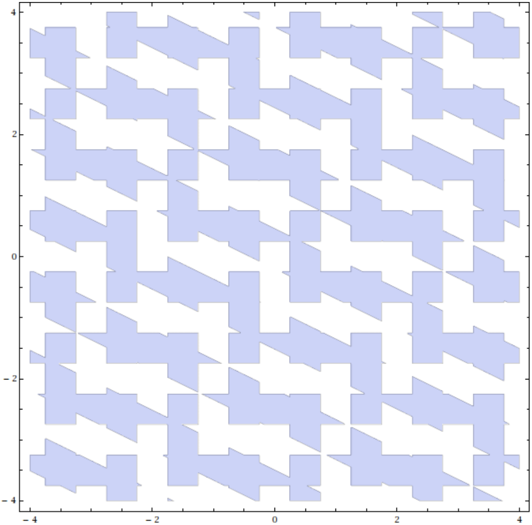

Planar section of the μ-cube orthogonal to $(372, 759, 1000)$

A topological invariant [Zorich, Dynnikov]

Let $H : \mathbb{T}^3 \to \mathbb{R}$ be a generic smooth function & $M_e = H^{-1}(e)$.
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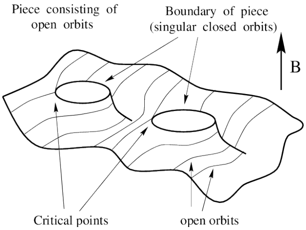

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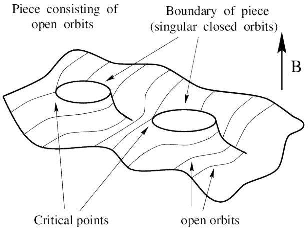

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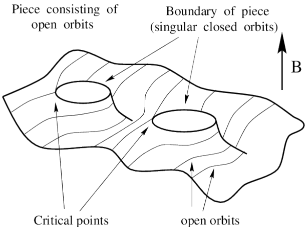

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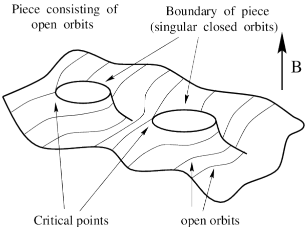

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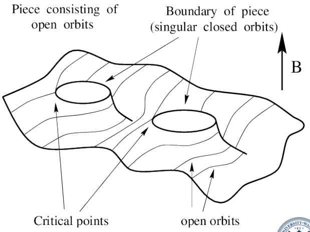

Note that ℓ dictates the aymptotics of open orbits:

if $B \in D_\ell$, then all open obits are strongly asymptotic to "$B \times \ell$".

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where $pr_1 : \mathbb{R}P^2 \times \mathbb{R} \to \mathbb{R}P^2$ with $pr_1(B,e) = B$.

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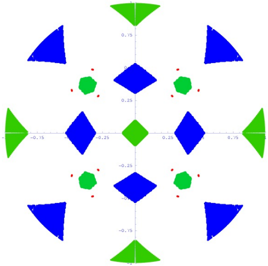
Examples

$$\mathcal{S}\mathcal{Z}\left(\qquad \text{(gold.u3d)} \qquad \right) =$$



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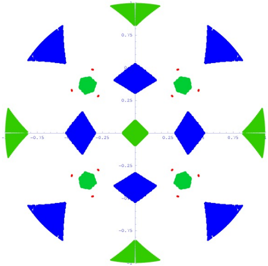
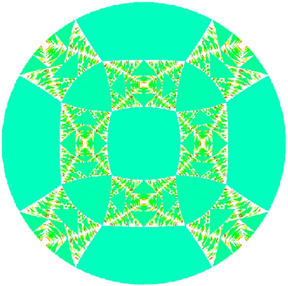


$$\mathcal{S}\mathcal{Z}\left(\cos x + \cos y + \cos z\right) =$$



Open Questions

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   - A. Avila, P. Hubert and A. Skripchenko, “Diffusion for chaotic plane sections of 3-periodic surfaces”, arXiv:1412.7913v2
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2. Given a $H : \mathbb{T}^3 \to \mathbb{R}$, what can be said about the Lebesgue measure and Hausdorff dimension of $\mathcal{E}(H)$?

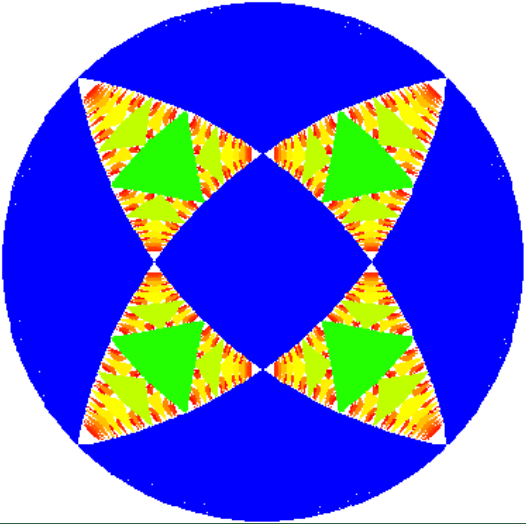
Conjecture (Novikov)

For every $H : \mathbb{T}^3 \to \mathbb{R}$,
$\mathcal{E}(H)$ has zero measure and $1 < \dim_H(\mathcal{E}(H)) < 2$.

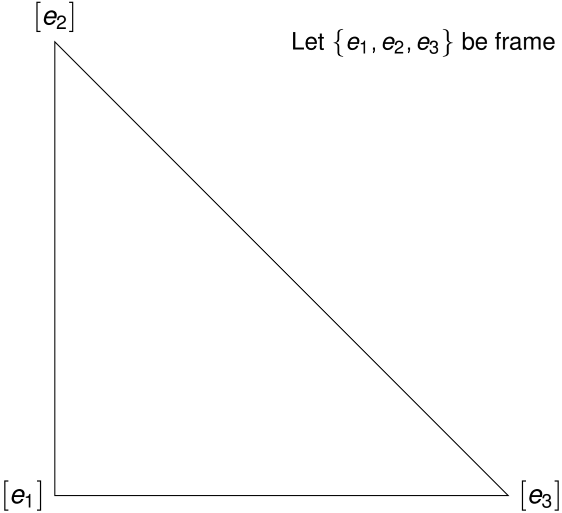
- R. De Leo and I. Dynnikov, "Geometry of plane sections of the infinite regular polyhedron $\{4,6|4\}$", Geom. Dedic. 138:1 (2009)
- A. Avila, P. Hubert and A. Skripchenko, "On the Hausdorff dimension of the Rauzy gasket", arXiv:1311.5361v2

The Fractal corresponding to $f_{\mu C}$

Because of symmetries, $\mathcal{SZ}(f_{\mu C}) = \mathcal{SZ}(\mu C) =$



The Fractal's Algorithm



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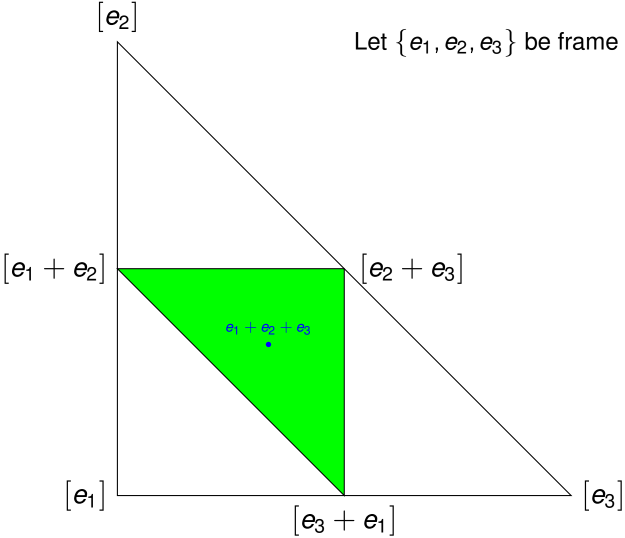

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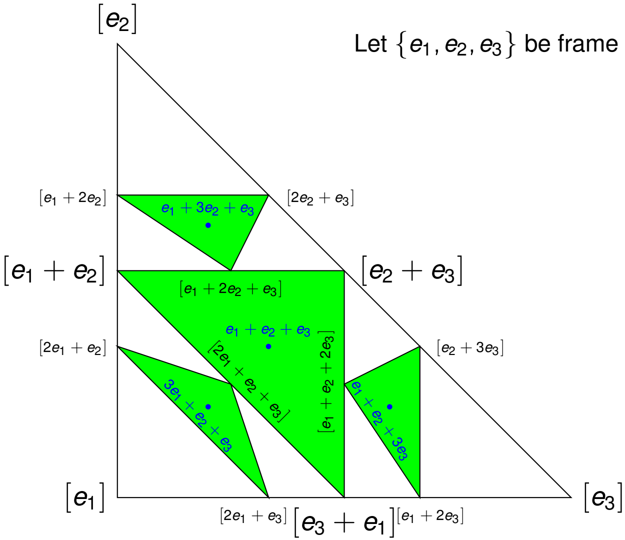

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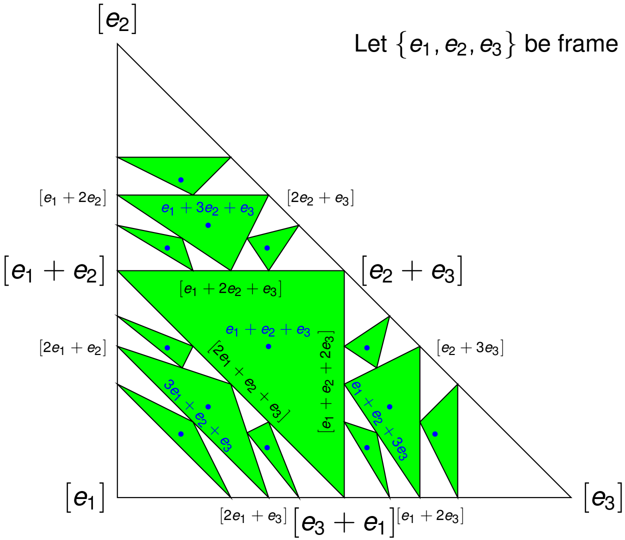

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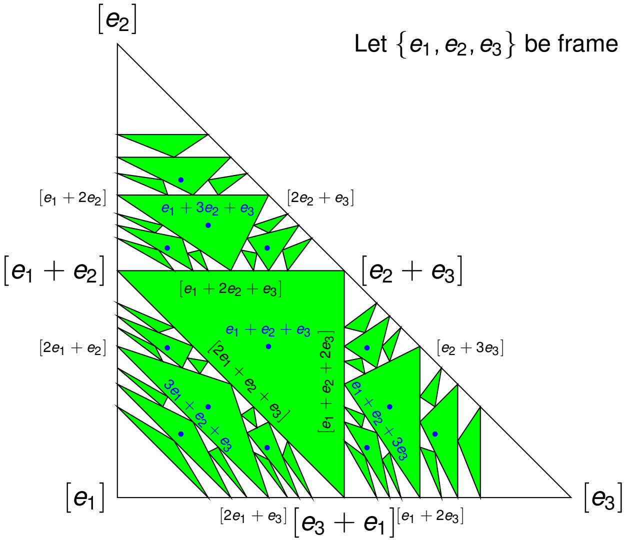

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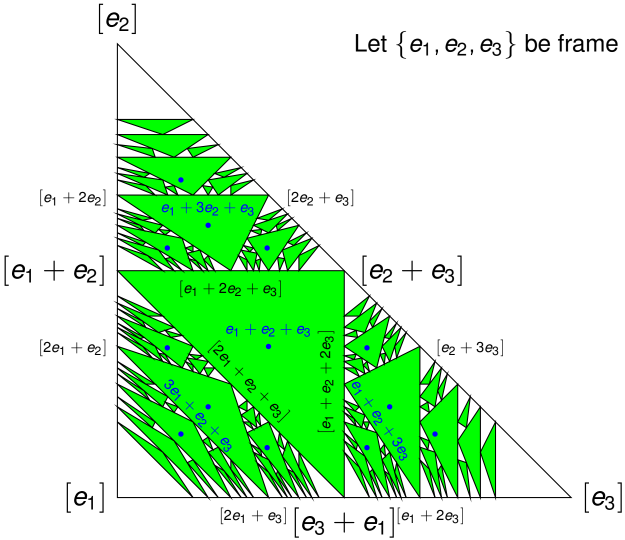

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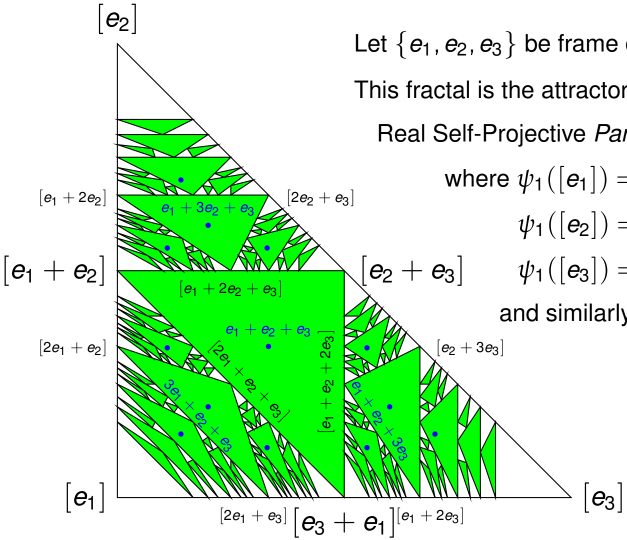

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This fractal is the attractor of the

Real Self-Projective *Parabolic* IFS $\{\psi_1, \psi_2, \psi_3\}$,

where $\psi_1([e_1]) = [e_1]$,

$\psi_1([e_2]) = [e_2 + e_1]$,

$\psi_1([e_3]) = [e_3 + e_1]$,

and similarly for ψ_2, ψ_3.

A first test for Novikov's Conjecture

To date, the fractal $\mathcal{E}(f_{\mu\boldsymbol{C}})$ is the only one of this class of fractals for which it is known an exact description, and so the first one against which the Novikov Conjecture can be tested.

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Theorem (Avila, Hubert, Skripchenko (2013))

$\dim_H \mathcal{E}(f_{\mu \boldsymbol{C}}) < 2$

It is not known yet whether $1 < \dim_H \mathcal{E}(f_{\mu \boldsymbol{C}})$.
Numerical evaluations[2] indicate that $\dim_H \mathcal{E}(f_{\mu \boldsymbol{C}}) \simeq 1.72$.

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Asymptotics of $\mathcal{E}(f_{\mu \boldsymbol{C}})$

The projective automorphisms ψ_1, ψ_2, ψ_3 used to build $\mathcal{E}(f_{\mu \boldsymbol{C}})$ are induced,w/resp to the frame $\{e_1, e_2, e_3\}$, by the linear maps

$$\boldsymbol{C} = \left\langle \underset{C_1}{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}, \underset{C_2}{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}}, \underset{C_3}{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}} \right\rangle \subset SL_3(\mathbb{N})$$

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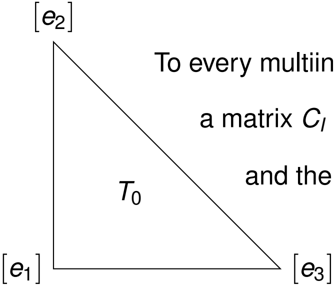

To every multiindex $I = i_1 \dots i_k$, $1 \leq i_r \leq 3$, correspond

a matrix $C_I = C_{i_1} \cdots C_{i_k}$, a triangle T_I

and the relative cut-out triangle Z_I.

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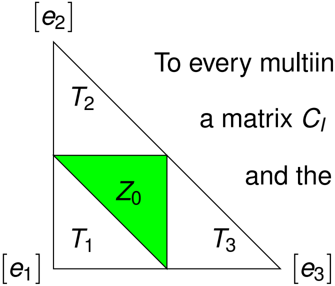

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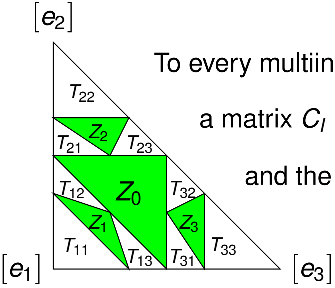

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Asymptotics of $\mathcal{E}(f_{\mu\boldsymbol{C}})$

The norms of the $C_l \in \langle C_1, C_2, C_3 \rangle$ determine the geometric asymptotics of the T_l and Z_l:

$$\frac{a}{\|C_l\|^3} \leq \text{Area}(T_l) \leq \frac{b}{\|C_l\|^2}$$

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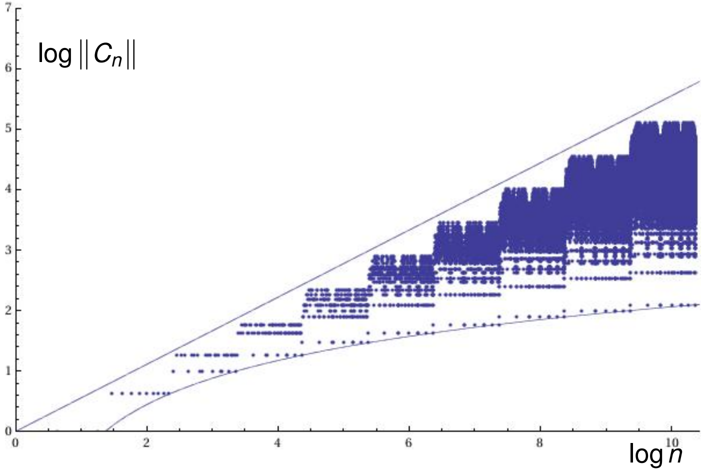
The asymptotic growth of $\|C_l\|$ is non-trivial, since some norms grow polynomially and some exponentially: e.g. $\|C_1^{2k}\| \propto k$ while $\|(C_1 C_2)^k\| \propto (1+g)^k$, where g is the Golden ratio.

$$C_1^{2k} = \begin{pmatrix} 1 & 0 & 0 \\ 2k & 1 & 0 \\ 2k & 0 & 1 \end{pmatrix}, \quad C_1 C_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

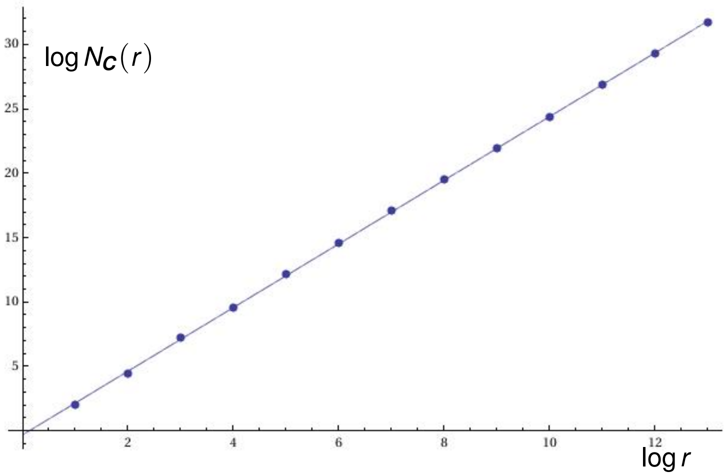
Norm Asymptotics in $\boldsymbol{C}$ – lexicographic order

Log-log plot of norms of elements of $\boldsymbol{C}$ in lexicographic order:



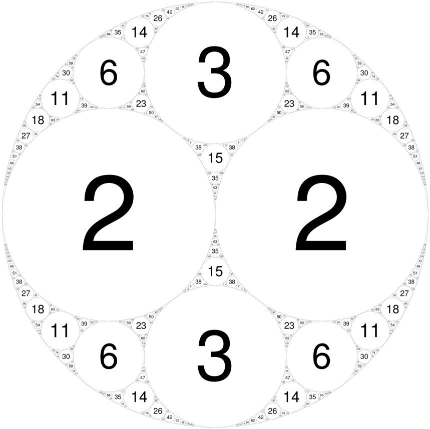
Norm Asymptotics in $\boldsymbol{C}$ – non-decreasing order

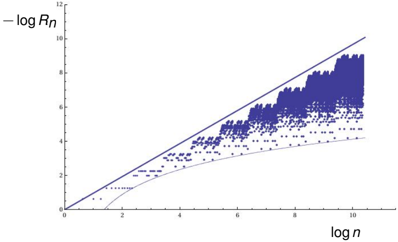
Log-log plot of norms of elements of $\boldsymbol{C}$ in non-decreasing order:

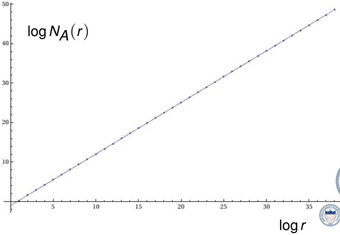


Radii Asymptotics in the Apollonian Gasket

Such behaviour is not uncommon, for example it is shared by the distribution of radii in celebrated Apollonian gasket:







Asymptotics of norms in semigroups of matrices

Theorem (RdL, 2012)

Let $\boldsymbol{S} = \langle A_i \rangle$ be a free finitely generated subsemigroup of $SL_n(K)$, $K = \mathbb{R}, \mathbb{C}$.
Under natural conditions (satisfied by $\boldsymbol{C}$ and the Apollonian semigroup),

$$\lim_{r\to\infty} \frac{\log N_{\boldsymbol{S}}(r)}{\log r} = s_{\boldsymbol{S}} < \infty.$$

Moreover,

$$s_{\boldsymbol{S}} = \sup_{s\geq 0}\{s|\sum_I \|A_I\|^{-s} = \infty\} = \inf_{s\geq 0}\{s|\sum_I \|A_I\|^{-s} < \infty\}.$$

A conjecture on the lower bound of $\dim_H \mathcal{E}(f_{\mu C})$

Given a free finitely generated semigroup $\boldsymbol{S} \subset SL_3(\mathbb{R})$, the corresponding semigroup of automorphisms of $\mathbb{R}P^2$ has, under some natural conditions, a unique compact invariant set.

In case of the semigroup $\boldsymbol{C}$, this set is exactly $\mathcal{E}(f_{\mu C})$.

[3] R. De Leo, *On the exponential growth of norms in semigroups of linear endomorphisms and the Hausdorff dimension of attractors of projective Iterated Function Systems*, to appear on *J. of Geometrical Analysis*

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Based on many numerical experiments and several analytical particular results, in a recent work[3] I formulated the following general conjecture:

Conjecture (RdL, 2012)

$(n+1)\dim_H R_{\boldsymbol{S}} \geq n s_{\boldsymbol{S}}$

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Since $s_{\boldsymbol{C}} \simeq 2.4438$, according to this conjecture $\dim_H \mathcal{E}(f_{\mu \boldsymbol{C}}) \geq 1.63$.

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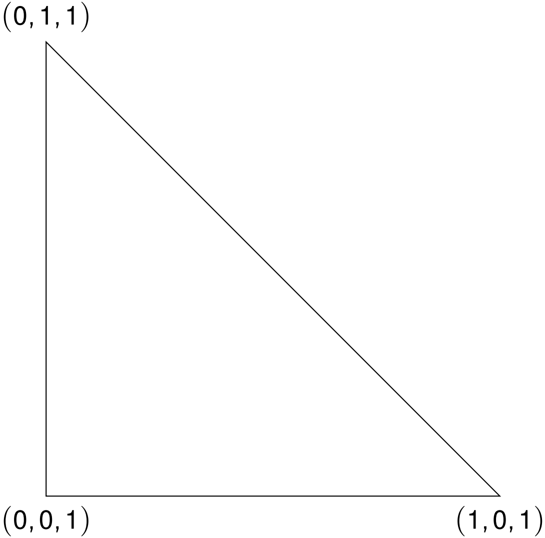
Tribonacci numbers in $\mathcal{E}(f_{\mu \boldsymbol{c}})$



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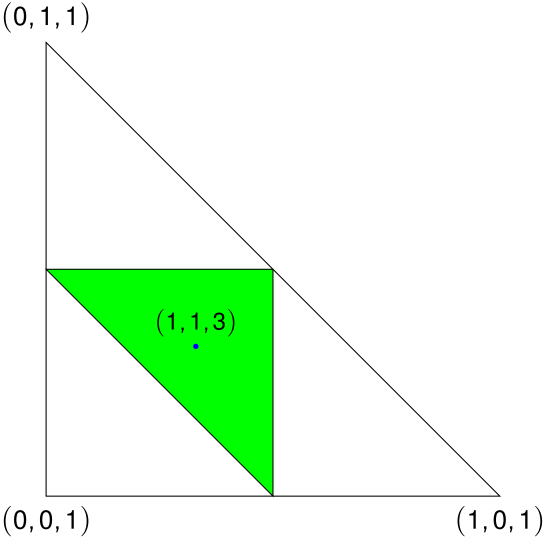

1 1 3

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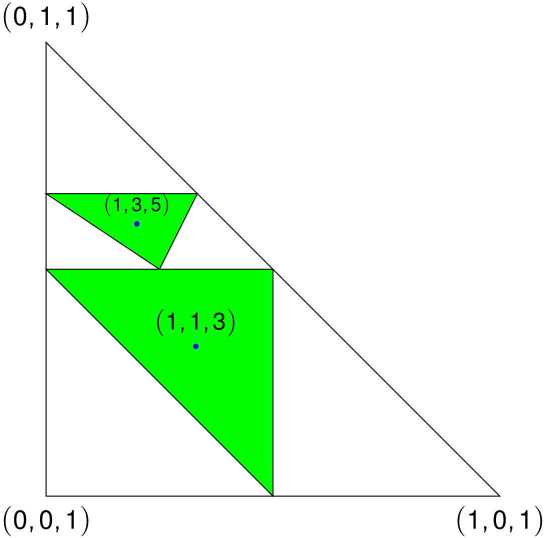

1	1	3
1	3	5

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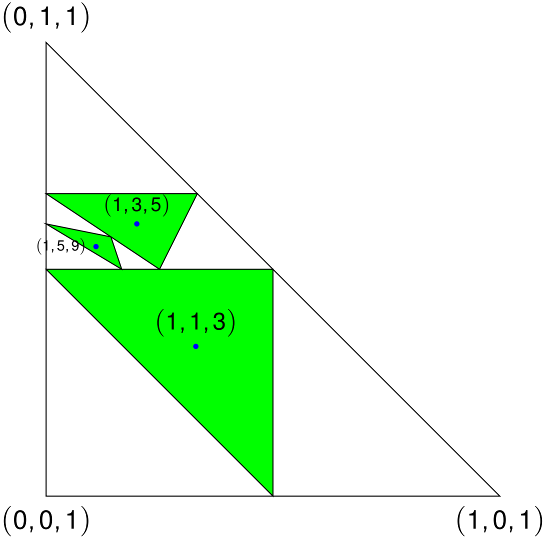

1	1	3
1	3	5
1	5	9

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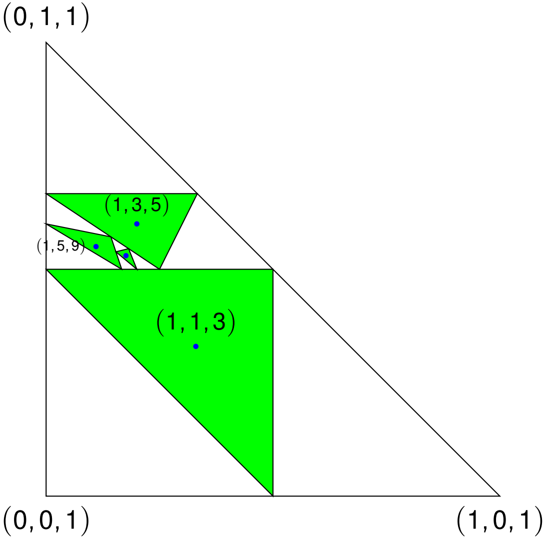

1	1	3
1	3	5
1	5	9
3	9	17

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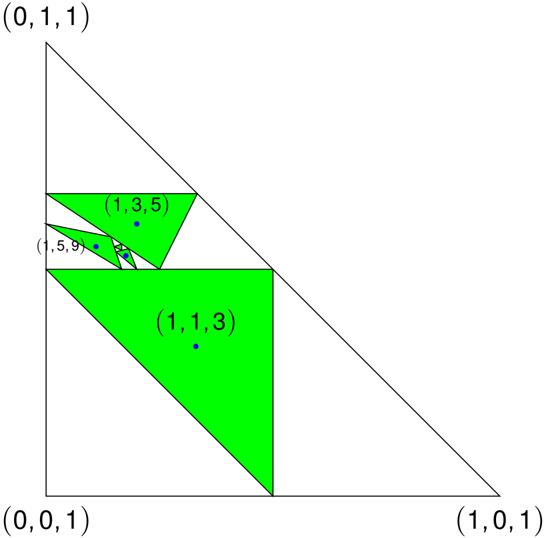
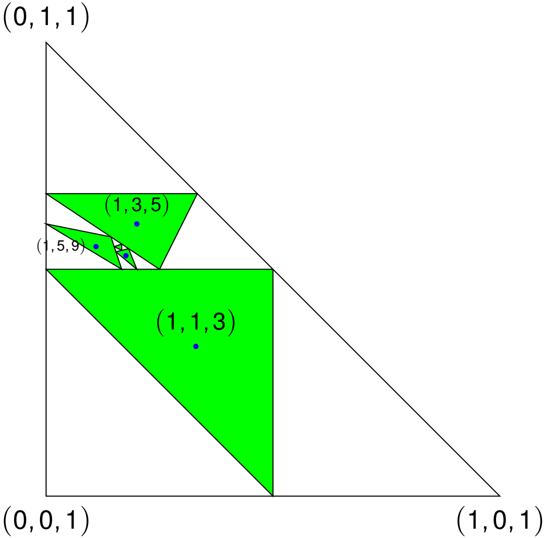


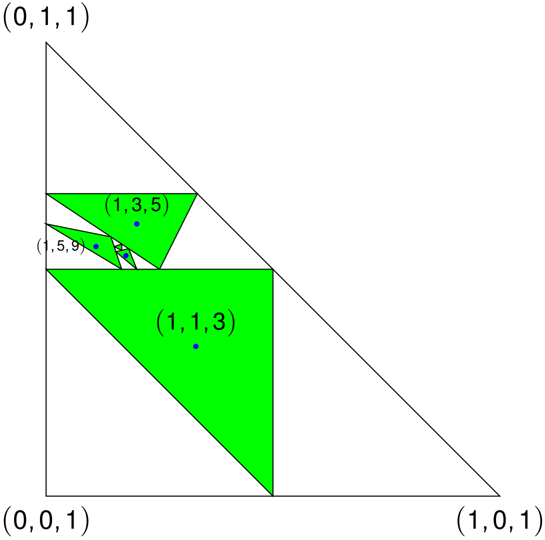
1	1	3
1	3	5
1	5	9
3	9	17
5	17	31

Tribonacci numbers in $\mathcal{E}(f_{\mu c})$



1	1	3
1	3	5
1	5	9
3	9	17
5	17	31
9	31	57

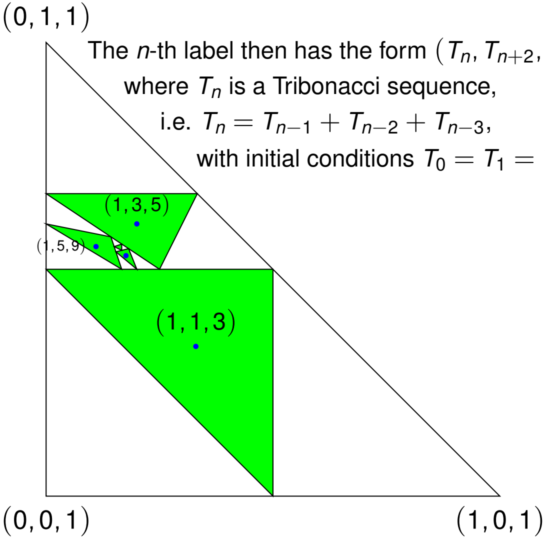
Tribonacci numbers in $\mathcal{E}(f_{\mu c})$



1	1	3
1	3	5
1	5	9
3	9	17
5	17	31
9	31	57
17	57	105

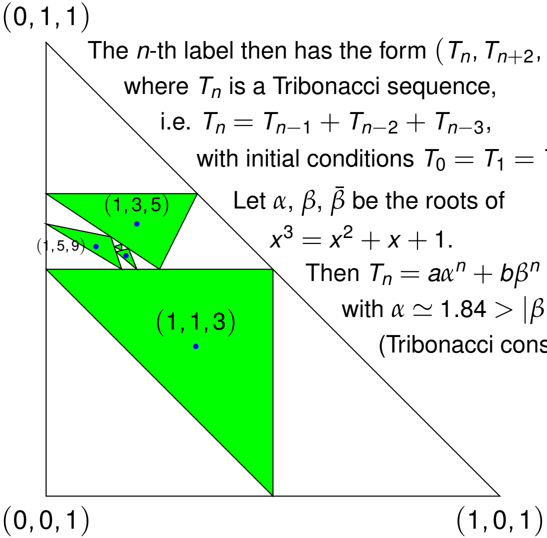
Tribonacci numbers in $\mathcal{E}(f_{\mu c})$

The n-th label then has the form (T_n, T_{n+2}, T_{n+3}) where T_n is a Tribonacci sequence, i.e. $T_n = T_{n-1} + T_{n-2} + T_{n-3}$, with initial conditions $T_0 = T_1 = T_2 = 1$.



1	1	3
1	3	5
1	5	9
3	9	17
5	17	31
9	31	57
17	57	105
	⋮	
T_n	T_{n+2}	T_{n+3}

Tribonacci numbers in $\mathcal{E}(f_{\mu c})$



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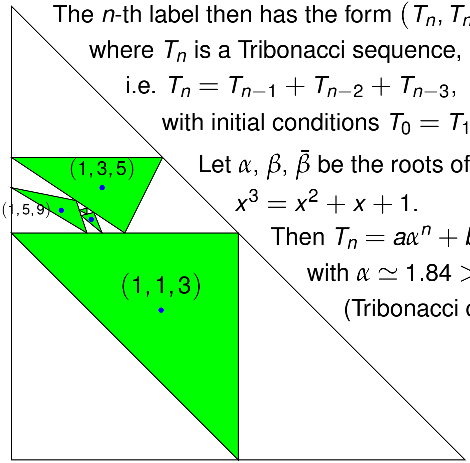
Let α, β, $\bar{\beta}$ be the roots of $x^3 = x^2 + x + 1$.

Then $T_n = a\alpha^n + b\beta^n + \bar{b}\bar{\beta}^n$ with $\alpha \simeq 1.84 > |\beta|$ (Tribonacci const.)

1	1	3
1	3	5
1	5	9
3	9	17
5	17	31
9	31	57
17	57	105
	⋮	
T_n	T_{n+2}	T_{n+3}

Tribonacci numbers in $\mathcal{E}(f_{\mu c})$



The n-th label then has the form (T_n, T_{n+2}, T_{n+3})

where T_n is a Tribonacci sequence,

i.e. $T_n = T_{n-1} + T_{n-2} + T_{n-3}$,

with initial conditions $T_0 = T_1 = T_2 = 1$.

Let α, β, $\bar\beta$ be the roots of

$x^3 = x^2 + x + 1$.

Then $T_n = a\alpha^n + b\beta^n + \bar b\bar\beta^n$

with $\alpha \simeq 1.84 > |\beta|$

(Tribonacci const.)

1	1	3
1	3	5
1	5	9
3	9	17
5	17	31
9	31	57
17	57	105
	⋮	
T_n	T_{n+2}	T_{n+3}
	↓	
1	α^2	α^3

3-irrational direction

$[1 : \alpha^2 : \alpha^3] \in E(f_{\mu c})$

Motivation: magnetoresistance in metals

In the so-called *semiclassical approximation*, the conductivity properties of a metal are encoded in the *Fermi Surface* $H(p_x, p_y, p_z) = E_F$, where $H(p)$ is the *Fermi Function*, triply periodic in its three arguments, and E_F the *Fermi energy*.

Under a constant magnetic field $B = (B_x, B_y, B_z)$, (quasi-)electrons' orbits are given by the intersection between the planes perpendicular to B and the Fermi Surface. In particular, the *magnetoresistance* of the metal is sensitive to the presence of open orbits of quasi-electrons, so that *it can be measured experimentally whether a given direction gives rise to open orbits* in case of the Fermi Surfaces of metals.

Example: the Fermi Surface of Gold

(gold.u3d)

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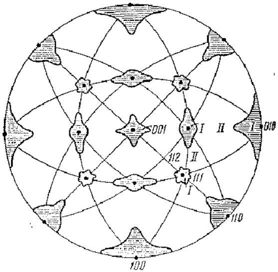
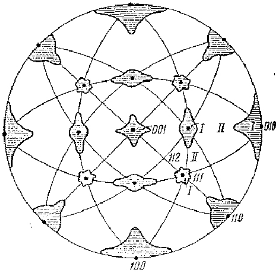
(gold.u3d)



From Yu. P. Gaidukov, *“Topology of the Fermi Surface for Gold”*, JETP 10 (1960)

Example: the Fermi Surface of Gold

(gold.u3d)



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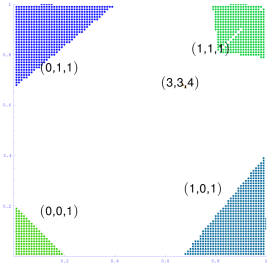
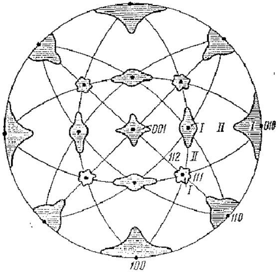


Example: the Fermi Surface of Gold

(gold.u3d)



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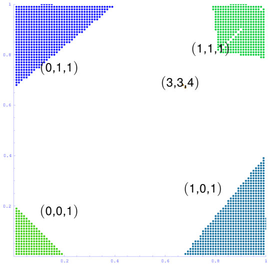
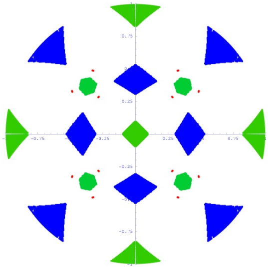
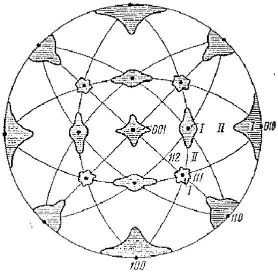




From R. De Leo, *"Topological effects in the magnetoresistance of Au and Ag"*, Phys. Lett. A (2004)

Example: the Fermi Surface of Gold

(gold.u3d)



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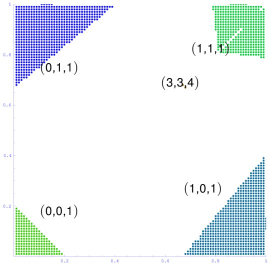

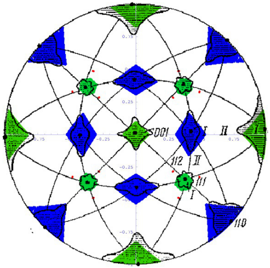

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