Partially Isometric Immersions and Free Maps

Roberto De Leo[1,2]

joint work with G. D'Ambra and A. Loi

[1]Dipartmento di Matematica
Università di Cagliari, Italy

[2]INFN, sez. di Cagliari, Italy

DGA 2010 - Brno, August 27-31

A Fundamental Question

A natural question in mathematics is whether an arbitrary "object" of a class defined abstractly can be seen as a "sub-object" of some "canonical" object of the class. For example:

Theorem (Whitney, 1944)

Every m-dimensional C^∞ manifold M admits an embedding into $\mathbb{R}^{2m}$ and an immersion into $\mathbb{R}^{2m-1}$.

A Fundamental Question

A natural question in mathematics is whether an arbitrary "object" of a class defined abstractly can be seen as a "sub-object" of some "canonical" object of the class. For example:

Theorem (Whitney, 1944)

Every m-dimensional C^∞ manifold M admits an embedding into $\mathbb{R}^{2m}$ and an immersion into $\mathbb{R}^{2m-1}$.

Theorem (Nash, 1956; Gromov, 1986)

Every m-dimensional C^∞ Riemannian manifold M admits an embedding into $\mathbb{R}^{m^2+5m+3}$.

Distributions

Our ultimate goal is to answer the analogue question for Riemannian distributions.

In more detail:

Definition

Let M be a C^∞ manifold. A *k-distribution* $\mathcal{H}$ on M is a vector subbundle $\mathcal{H} \subset TM$ such that $\dim \mathcal{H}_m = k$, $\forall m \in M$.

Distributions

Our ultimate goal is to answer the analogue question for Riemannian distributions.

In more detail:

Definition

Let M be a C^∞ manifold. A *k-distribution* $\mathcal{H}$ on M is a vector subbundle $\mathcal{H} \subset TM$ such that $\dim \mathcal{H}_m = k$, $\forall m \in M$.

Example

A vector field ξ on M with no singular point determines a 1-distribution $\mathcal{H} = span\, \xi$.
A 1-form ω on M with no singular point determines a "codimension-1"-distribution $\mathcal{H} = \ker \omega$.

Riemannian distributions

Definition

A *Riemannian k-distribution* $(\mathcal{H}, g)$ on M is a pair of a k-distribution $\mathcal{H} \subset TM$ and a positive-definite section g of the symmetric tendor product of $\mathcal{H}^*$ by itself.

Riemannian distributions

Definition

A *Riemannian k-distribution* $(\mathcal{H}, g)$ on M is a pair of a k-distribution $\mathcal{H} \subset TM$ and a positive-definite section g of the symmetric tendor product of $\mathcal{H}^*$ by itself.

Example

- The restriction to $\mathcal{H}$ of any Riemannian metric on M makes $\mathcal{H}$ Riemannian.
- If the vector field ξ on M has no singular point, every strictly positive function $\psi : M \longrightarrow \mathbb{R}$ induces a metric on any 1-dimensional distribution $\mathcal{H} = span\, \xi$ by setting $g(\xi_x, \xi_x) = \psi(x)$.

$\mathcal{H}$-immersions

Definition

Let g be a metric on $\mathcal{H}$, i.e. a positive-definite symmetric section of $\mathcal{H}^* \otimes \mathcal{H}^*$. We say that $f \in C^\infty(M, \mathbb{R}^q)$ is a $\mathcal{H}$-immersion of M into $\mathbb{R}^q$ if $f^* e_q|_{\mathcal{H}} = g$.

Clearly $\mathcal{H}$-immersion can exist only for $q \geq k$.

$\mathcal{H}$-immersions

Definition

Let g be a metric on $\mathcal{H}$, i.e. a positive-definite symmetric section of $\mathcal{H}^* \otimes \mathcal{H}^*$. We say that $f \in C^\infty(M, \mathbb{R}^q)$ is a $\mathcal{H}$-immersion of M into $\mathbb{R}^q$ if $f^* e_q|_{\mathcal{H}} = g$.

Clearly $\mathcal{H}$-immersion can exist only for $q \geq k$.

Example

Let ξ a vector field without zeros on M, $\mathcal{H} = span\{\xi\}$ the corresponding 1-distribution on M, f a map $M \longrightarrow \mathbb{R}$ and θ a base of $\mathcal{H}^*$ dual to ξ (so that $\theta(\xi) = 1$). Then

$$f^* e_q\big|_{\mathcal{H}} = (L_\xi f)^2 \theta \otimes \theta \,.$$

Hence f is an $\mathcal{H}$-immersion iff either $L_\xi f > 0$ or $L_\xi f < 0$.

$\mathcal{H}$-immersions

Example

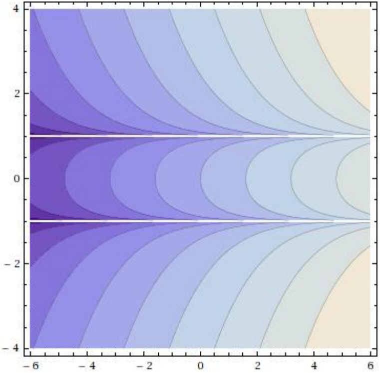
Consider the distribution $\mathcal{H}_\xi$ associated to the planar vector field $\xi(x, y) = 2y\partial_x + (1 - y^2)\partial_y$.

$\mathcal{H}_\xi$-immersions may arise for $q \geq 1$ and $\mathcal{D}_{\mathcal{H}_\xi,1}(f) = (L_\xi f)^2 \theta \otimes \theta$, so a map $f : \mathbb{R}^2 \to \mathbb{R}$ is a $\mathcal{H}_\xi$-immersions if $L_\xi f \neq 0$ on $\mathbb{R}^2$.

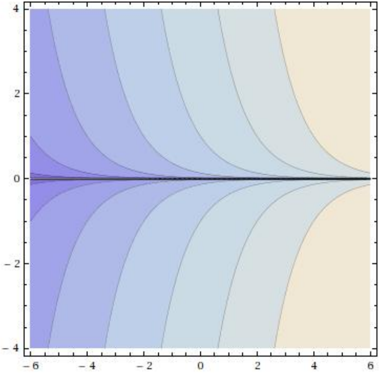
Topologically, this is equivalent to the fact that the integral trajectories of ξ are everywhere transversal to f's level sets.

E.g. $f(x, y) = ye^x$ is a $\mathcal{H}_\xi$-immersion of $\mathbb{R}^2$ into $\mathbb{R}$ since $L_\xi f(x, y) = (1 + y^2)e^x > 0$.

$\mathcal{H}$-immersions



Integral Trajectories
of $\xi(x,y) = 2y\partial_x + (1-y^2)\partial_y$



Level sets
of $f(x,y) = ye^x$

Ultimate goal and Intermediate results

Question

Can every Riemannian k-distribution on M be induced via some $\mathcal{H}$-immersion $f : M \longrightarrow \mathbb{R}^q$?

We believe the answer is positive.

Ultimate goal and Intermediate results

Question

Can every Riemannian k-distribution on M be induced via some $\mathcal{H}$-immersion $f : M \longrightarrow \mathbb{R}^q$?

We believe the answer is positive.

So far we rather focused on two simpler tasks:

1. Solve a weaker version of our ultimate goal: if some metric g on $\mathcal{H}$ is induced via some $f : M \longrightarrow \mathbb{R}^q$ and we slightly deform g into g', can we deform f into f' so that f' induces g' on $\mathcal{H}$?

Ultimate goal and Intermediate results

Question

Can every Riemannian k-distribution on M be induced via some $\mathcal{H}$-immersion $f : M \to \mathbb{R}^q$?

We believe the answer is positive.

So far we rather focused on two simpler tasks:

1. Solve a weaker version of our ultimate goal: if some metric g on $\mathcal{H}$ is induced via some $f : M \to \mathbb{R}^q$ and we slightly deform g into g', can we deform f into f' so that f' induces g' on $\mathcal{H}$?
2. Find concrete cases of "geometrically interesting" distributions where this happens for the lowest q possible (i.e. in "critical dimension").

The Isometric Operator

$Imm(M, \mathbb{R}^q)$ immersions of the smooth manifold M into $\mathbb{R}^q$

e_q Euclidean metric on $\mathbb{R}^q$

$S_2^0(M)$ symmetric tensor product of T^*M by itself

$\Gamma^\infty(S_2^0(M))$ smooth sections of $S_2^0(M) \to M$

$\mathcal{G}(M) \subset \Gamma^\infty(S_2^0(M))$ set of Riemannian metrics over M

Definition

We call *isometric operator* the map

$$\begin{array}{cccc} \mathcal{D}_{M,q}: & Imm(M, \mathbb{R}^q) & \longrightarrow & \mathcal{G}(M) \\ & f & \longmapsto & f^* e_q \end{array}$$

The Isometric Operator

- $\mathcal{D}_{M,q}$ is a continuous map when the functional spaces are endowed with Whitney strong topology.

The Isometric Operator

- $\mathcal{D}_{M,q}$ is a continuous map when the functional spaces are endowed with Whitney strong topology.
- $\mathcal{D}_{M,q}$ is central in the isometric immersions theory: e.g. every m-dimensional Riemannian manifold can be isometrically immersed into $\mathbb{R}^q$ iff all operators $\mathcal{D}_{M,q}$, $\dim M = m$, are surjective.

The Isometric Operator

- $\mathcal{D}_{M,q}$ is a continuous map when the functional spaces are endowed with Whitney strong topology.
- $\mathcal{D}_{M,q}$ is central in the isometric immersions theory: e.g. every m-dimensional Riemannian manifold can be isometrically immersed into $\mathbb{R}^q$ iff all operators $\mathcal{D}_{M,q}$, $\dim M = m$, are surjective.
- **The property we focus on here is the following weaker version: is $\mathcal{D}_{M,q}$, restricted to some open subset of $Imm(M, \mathbb{R}^q)$, an open map? In other words, if**

$$\mathcal{D}_{M,q}(f_0) = g_0$$

then is $\mathcal{D}_{M,q}(f) = g$ is solvable in some nbhd of g_0?

Linearization of the Isometric Operator

Use indices $\alpha, \beta = 1, \ldots, m$ and $i, j = 1, \ldots, q$.
In coords (x^α) on M and (y^i) on $\mathbb{R}^q$
f writes as $(f^i(x^\alpha))$ and the equation $\mathcal{D}(f) = g$ writes as

$$\boxed{\delta_{ij}\partial_\alpha f^i \partial_\beta f^j = g_{\alpha\beta}}$$

Linearization of the Isometric Operator

Use indices $\alpha, \beta = 1, \ldots, m$ and $i, j = 1, \ldots, q$.
In coords (x^α) on M and (y^i) on $\mathbb{R}^q$
f writes as $(f^i(x^\alpha))$ and the equation $\mathcal{D}(f) = g$ writes as

$$\boxed{\delta_{ij}\partial_\alpha f^i \partial_\beta f^j = g_{\alpha\beta}}$$

Take a smooth curve g_λ and look for a solution f_λ. Then

$$\boxed{2\delta_{ij}\partial_\alpha f^i \partial_\beta \delta f^j = \delta g_{\alpha\beta}}$$

where $\delta f^i = \frac{df^i_\lambda}{d\lambda}\Big|_{\lambda=0}$ and $\delta g_{\alpha\beta} = \frac{d(g_\lambda)_{\alpha\beta}}{d\lambda}\Big|_{\lambda=0}$.

Linearization of the Isometric Operator

Definition

The operator $\ell_{M,q}(f,\delta f) = 2\delta_{ij}\partial_\alpha f^i \partial_\beta \delta f^j$ is called the *linearization* of $\mathcal{D}_{M,q}$. We say that $\mathcal{D}_{M,q}$ is *infinitesimally invertible* over some open set $\mathcal{A} \subset Imm(M,\mathbb{R}^q)$ if the equation

$$\boxed{\ell_{M,q}(f,\delta f) = \delta g}$$

is solvable over $\mathcal{A}$.

Linearization of the Isometric Operator

Definition

The operator $\ell_{M,q}(f,\delta f) = 2\delta_{ij}\partial_\alpha f^i \partial_\beta \delta f^j$ is called the *linearization* of $\mathcal{D}_{M,q}$. We say that $\mathcal{D}_{M,q}$ is *infinitesimally invertible* over some open set $\mathcal{A} \subset Imm(M, \mathbb{R}^q)$ if the equation

$$\boxed{\ell_{M,q}(f,\delta f) = \delta g}$$

is solvable over $\mathcal{A}$.

Theorem (Newton-Nash-Moser-Gromov IFT)

If $\mathcal{D}_{M,q}$ is infinitesimally invertible over $\mathcal{A}$, then

$$\mathcal{D}_{M,q}|_{\mathcal{A}} : \mathcal{A} \to \mathcal{G}(M)$$

is an open map.

Nash's trick

Using Nash's trick the linearized equation

$$2\delta_{ij}\partial_\alpha f^i \partial_\beta \delta f^j = \delta g_{\alpha\beta}$$

becomes a fully algebraic system
of $q_m := m + m(m+1)/2$ eqs. in the q unknowns δf^i:

$$\begin{cases} \delta_{ij}\partial_\alpha f^i \delta f^j = h_\alpha \\ \delta_{ij}\partial_{\alpha\beta} f^i \delta f^j = \partial_\alpha h_\beta + \partial_\beta h_\alpha - \delta g_{\alpha\beta}/2 \end{cases}$$

Nash's trick

Using Nash's trick the linearized equation

$$\boxed{2\delta_{ij}\partial_\alpha f^i \partial_\beta \delta f^j = \delta g_{\alpha\beta}}$$

becomes a fully algebraic system
of $q_m := m + m(m+1)/2$ eqs. in the q unknowns δf^i:

$$\boxed{\begin{cases} \delta_{ij}\partial_\alpha f^i \delta f^j = h_\alpha \\ \delta_{ij}\partial_{\alpha\beta} f^i \delta f^j = \partial_\alpha h_\beta + \partial_\beta h_\alpha - \delta g_{\alpha\beta}/2 \end{cases}}$$

Clearly a sufficient condition for the solvability of this system is that the $q_m \times q$ matrix $D_2 f = \begin{pmatrix} \partial_\alpha f^i \\ \partial_{\alpha\beta} f^i \end{pmatrix}$ have rank q_m.

Free Maps

Definition

A map $f \in C^\infty(M, \mathbb{R}^q)$ s.t. $rk\, D_2 f = q_m$ at every point of M is called a *free map*. We denote by $Free(M, \mathbb{R}^q)$ the set of all smooth free maps $M \to \mathbb{R}^q$.

Free Maps

Definition

A map $f \in C^\infty(M, \mathbb{R}^q)$ s.t. $rk\, D_2 f = q_m$ at every point of M is called a *free map*. We denote by $Free(M, \mathbb{R}^q)$ the set of all smooth free maps $M \longrightarrow \mathbb{R}^q$.

As a corollary of the Newton-Nash-Moser-Gromov IFT
we get immediately that

Theorem (Nash, 1956)

The restriction of $\mathcal{D}_{M,q}$ to $Free(M, \mathbb{R}^q) \subset Imm(M, \mathbb{R}^q)$ is an open map.

Free Maps

- Clearly $Free(M, \mathbb{R}^q) = \emptyset$ if $q < q_m$ (we say that q_m is the "critical dimension").

Free Maps

- Clearly $Free(M, \mathbb{R}^q) = \emptyset$ if $q < q_m$ (we say that q_m is the "critical dimension").
- A standard transversality argument shows that being free is a generic property for $q \geq m + q_m$.

Free Maps

- Clearly $Free(M, \mathbb{R}^q) = \emptyset$ if $q < q_m$ (we say that q_m is the "critical dimension").
- A standard transversality argument shows that being free is a generic property for $q \geq m + q_m$.
- For $q_m \leq q < m + q_m$ the existence of free maps is a more complicated matter. E.g. if M is parallelizable we know that $Free(M, \mathbb{R}^q) \neq \emptyset$ for $q \geq q_m$ if M is open and for $q > q_m$ if M is closed (Gromov, Eliashberg).

Free Maps

- Clearly $Free(M, \mathbb{R}^q) = \emptyset$ if $q < q_m$ (we say that q_m is the "critical dimension").
- A standard transversality argument shows that being free is a generic property for $q \geq m + q_m$.
- For $q_m \leq q < m + q_m$ the existence of free maps is a more complicated matter. E.g. if M is parallelizable we know that $Free(M, \mathbb{R}^q) \neq \emptyset$ for $q \geq q_m$ if M is open and for $q > q_m$ if M is closed (Gromov, Eliashberg).

Example

The map $F(x^1, \dots, x^m) = (x^1, \dots, x^m, (x^1)^2, x^1x^2, \dots, (x^m)^2)$ belongs to $Free(\mathbb{R}^m, \mathbb{R}^{q_m})$.

Partial Isometries

Definition

We call *partially isometric operator* the map

$$\begin{array}{rccc} \mathcal{D}_{\mathcal{H},q}: & Imm_{\mathcal{H}}(M,\mathbb{R}^q) & \longrightarrow & \mathcal{G}(\mathcal{H}) \\ & f & \mapsto & f^* e_q|_{\mathcal{H}} \end{array}$$

Locally every k-distribution $\mathcal{H}$ on M
is the span of k vector fields ξ_a and

$$\boxed{\mathcal{D}_{\mathcal{H},q}(f) = \delta_{ij} L_{\xi_a} f^i L_{\xi_b} f^j \; \theta^a \otimes \theta^b}$$

where $L_{\xi_a} f^i$ is the Lie derivative of f^i with respect to ξ_a
and the 1-forms θ^a, $a = 1, \ldots, k$, are dual of the ξ_a in $\mathcal{H}^*$.

Linearization of $\mathcal{D}_{\mathcal{H},q}$

In a trivialization of $\mathcal{H}$ the equation $\mathcal{D}_{\mathcal{H},q}(f) = g$ writes as

$$\boxed{\delta_{ij} L_{\xi_a} f^i L_{\xi_b} f^j = g_{ab}}$$

Linearization of $\mathcal{D}_{\mathcal{H},q}$

In a trivialization of $\mathcal{H}$ the equation $\mathcal{D}_{\mathcal{H},q}(f) = g$ writes as

$$\boxed{\delta_{ij} L_{\xi_a} f^i L_{\xi_b} f^j = g_{ab}}$$

Take a smooth curve g_λ and look for a solution f_λ. Then

$$\boxed{2\delta_{ij} L_{\xi_a} f^i L_{\xi_b} \delta f^j = \delta g_{ab}}$$

where $\delta f^i = \frac{df^i_\lambda}{d\lambda}\Big|_{\lambda=0}$ and $\delta g_{ab} = \frac{d(g_\lambda)_{ab}}{d\lambda}\Big|_{\lambda=0}$.

Linearization of $\mathcal{D}_{\mathcal{H},q}$

By the Newton-Nash-Moser-Gromov IFT we know that a sufficient condition for $\mathcal{D}_{\mathcal{H},q}$ to be an open map over some open set $\mathcal{A} \subset C^\infty(M, \mathbb{R}^q)$ is that the linearized equation

$$\boxed{2\delta_{ij} L_{\xi_a} f^i L_{\xi_b} \delta f^j = g_{ab}}$$

be solvable for all $f \in \mathcal{A}$.

Linearization of $\mathcal{D}_{\mathcal{H},q}$

By the Newton-Nash-Moser-Gromov IFT we know that a sufficient condition for $\mathcal{D}_{\mathcal{H},q}$ to be an open map over some open set $\mathcal{A} \subset C^\infty(M, \mathbb{R}^q)$ is that the linearized equation

$$2\delta_{ij} L_{\xi_a} f^i L_{\xi_b} \delta f^j = g_{ab}$$

be solvable for all $f \in \mathcal{A}$.

Using the Nash trick we get the following equivalent fully algebraic system of $q_k = k + k(k+1)/2$ eqs in q unknowns δf^i:

$$\begin{cases} \delta_{ij} L_{\xi_a} f^i \delta f^j = h_a \\ \delta_{ij}(L_{\xi_a} L_{\xi_b} + L_{\xi_b} L_{\xi_a}) f^i \delta f^j = L_{\xi_a} h_b + L_{\xi_b} h_a - \delta g_{ab}/2 \end{cases}$$

$\mathcal{H}$-free maps

Clearly a sufficient condition for the resolution of such system is that the $q_k \times q$ matrix

$$D_{\xi_1,\cdots,\xi_k}(f) = \begin{pmatrix} L_{\xi_1} f^1 & \cdots & L_{\xi_1} f^q \\ \vdots & \vdots & \vdots \\ L_{\xi_k} f^1 & \cdots & L_{\xi_k} f^q \\ L^2_{\xi_1} f^1 & \cdots & L^2_{\xi_1} f^q \\ L_{\xi_1} L_{\xi_2} f^1 + L_{\xi_2} L_{\xi_1} f^1 & \cdots & L_{\xi_1} L_{\xi_2} f^q + L_{\xi_2} L_{\xi_1} f^q \\ \vdots & \vdots & \vdots \\ L^2_{\xi_k} f^1 & \cdots & L^2_{\xi_k} f^q \end{pmatrix}$$

have rank q_k at every point.

$\mathcal{H}$-free maps

Definition (Gromov)

A $\mathcal{H}$-immersion f of M into $\mathbb{R}^q$ s.t. $rk\, D_{\xi_1,\cdots,\xi_k} f = q_k$ for every trivialization of $\mathcal{H}$ at every point of M is called a *$\mathcal{H}$-free map*. We denote by $Free_{\mathcal{H}}(M, \mathbb{R}^q)$ the set of all smooth $\mathcal{H}$-free maps $M \to \mathbb{R}^q$.

$\mathcal{H}$-free maps

Definition (Gromov)

A $\mathcal{H}$-immersion f of M into $\mathbb{R}^q$ s.t. $rk\, D_{\xi_1,\cdots,\xi_k} f = q_k$ for every trivialization of $\mathcal{H}$ at every point of M is called a *$\mathcal{H}$-free map*. We denote by $Free_{\mathcal{H}}(M, \mathbb{R}^q)$ the set of all smooth $\mathcal{H}$-free maps $M \to \mathbb{R}^q$.

As a corollary of the Newton-Nash-Moser-Gromov IFT we get immediately that

Theorem 1

The restriction of $\mathcal{D}_{\mathcal{H},q}$ to $Free_{\mathcal{H}}(M, \mathbb{R}^q) \subset Imm_{\mathcal{H}}(M, \mathbb{R}^q)$ is an open map.

$\mathcal{H}$-free Maps

- Clearly $Free_{\mathcal{H}}(M, \mathbb{R}^q) = \emptyset$ if $q < q_k$ (we call q_k the "critical dimension").

$\mathcal{H}$-free Maps

- Clearly $Free_{\mathcal{H}}(M, \mathbb{R}^q) = \emptyset$ if $q < q_k$ (we call q_k the "critical dimension").
- A standard transversality argument shows that being free is a generic property for $q \geq m + q_k$.

$\mathcal{H}$-free Maps

- Clearly $Free_{\mathcal{H}}(M, \mathbb{R}^q) = \emptyset$ if $q < q_k$ (we call q_k the "critical dimension").
- A standard transversality argument shows that being free is a generic property for $q \geq m + q_k$.
- We did not investigate yet what happens in the intermediate range $q_k \leq q < m + q_k$.

$\mathcal{H}$-free maps of 1-distributions in $\mathbb{R}^2$ in critical dim.

Example

Consider again the case of a 1-distribution $\mathcal{H}_\xi = span\,\xi$.
$\mathcal{H}_\xi$-free maps can arise only for $q \geq 1 + q_1 = 2$.
A map $f = (f^1, f^2) : \mathbb{R}^2 \to \mathbb{R}^2$ is $\mathcal{H}_\xi$-free iff

$$D_\xi f = \begin{pmatrix} L_\xi f^1 & L_\xi f^2 \\ L_\xi^2 f^1 & L_\xi^2 f^2 \end{pmatrix}$$

has non-zero determinant at every point.

$\mathcal{H}$-free maps

Theorem 2

Let $\mathcal{H}$ be a k-distribtion on M. If $f \in Imm_{\mathcal{H}}(M, \mathbb{R}^k)$ and $F \in Free(\mathbb{R}^k, \mathbb{R}^{q_k})$ then $F \circ f \in Free_{\mathcal{H}}(M, \mathbb{R}^k)$.

$\mathcal{H}$-free maps

Proof.

The proof is just a direct calc., here we show only the case $k = 1$. Let $F = (\chi, \phi) \in Free(M, \mathbb{R}^2)$ and $f \in Imm_{\mathcal{H}}(M, \mathbb{R})$. Then

$$\det D_\xi F(f) = \begin{vmatrix} L_\xi[\chi(f)] & L_\xi[\phi(f)] \\ L_\xi^2[\chi(f)] & L_\xi^2[\phi(f)] \end{vmatrix} =$$

$$= \begin{vmatrix} \chi'(f) L_\xi f & \phi'(f) L_\xi f \\ \chi'(f) L_\xi^2 f + \chi''(f)[L_\xi f]^2 & \phi'(f) L_\xi^2 f + \phi''(f)[L_\xi f]^2 \end{vmatrix} =$$

$$= \begin{vmatrix} \chi'(f) & \phi'(f) \\ \chi''(f) & \phi''(f) \end{vmatrix} [L_\xi f]^3$$

□

Some classes of distr. $\mathcal{H}$ with $\mathcal{H}$-free maps in crit. dim.

Lemma (Weiner, 1988)

Let $\mathcal{H} \subset T\mathbb{R}^2$ a Hamiltonian distribution (i.e. $\mathcal{H} = \ker dH$ for some regular function $H : \mathbb{R}^2 \to \mathbb{R}$) and ξ any section of $\mathcal{H} \to M$ without zeros. Then there exists a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ s.t. $L_\xi f > 0$ (i.e. $Imm_{\mathcal{H}}(\mathbb{R}^2, \mathbb{R}) \neq \emptyset$).

Some classes of distr. $\mathcal{H}$ with $\mathcal{H}$-free maps in crit. dim.

Lemma (Weiner, 1988)

Let $\mathcal{H} \subset T\mathbb{R}^2$ a Hamiltonian distribution (i.e. $\mathcal{H} = \ker dH$ for some regular function $H : \mathbb{R}^2 \to \mathbb{R}$) and ξ any section of $\mathcal{H} \to M$ without zeros. Then there exists a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ s.t. $L_\xi f > 0$ (i.e. $Imm_{\mathcal{H}}(\mathbb{R}^2, \mathbb{R}) \neq \emptyset$).

Proposition 1

Under the hypothesis above, $Free_{\mathcal{H}}(\mathbb{R}^2, \mathbb{R}^2) \neq \emptyset$.

$\mathcal{H}$-free maps of 1-distributions in $\mathbb{R}^2$ in critical dimension

Example

Consider again $\xi = 2y\partial_x + (1-y^2)\partial_y$.
Recall that $L_\xi(ye^x) > 0$ and $\psi(t) = (t, t^2)$, $\hat{\psi}(t) = (\cos t, \sin t)$ belong to $Free(\mathbb{R}, \mathbb{R}^2)$ since $D_2\psi = \begin{pmatrix} 1 & 2x \\ 0 & 2 \end{pmatrix}$ and $D_2\hat{\psi} = \begin{pmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{pmatrix}$. Hence, for example, the maps

$$f(x,y) = (ye^x, y^2e^{2x})$$

and

$$\hat{f}(x,y) = (\cos(ye^x), \sin(ye^x))$$

both belong to $Free_{\mathcal{H}_\xi}(\mathbb{R}^2, \mathbb{R}^2)$.
Note that these are $\mathcal{H}_\xi$-free maps in critical dimension.

Some classes of distr. $\mathcal{H}$ with $\mathcal{H}$-free maps in crit. dim.

We generalized Weiner's Lemma in three different directions.
In the first we do not require ξ to be Hamiltonian:

Lemma (1. 1-distributions in $\mathbb{R}^2$)

Let ξ a vector field on $\mathbb{R}^2$ of finite type (i.e. the set of its separatrices is closed and each leaf is inseparable from just finitely many other leaves) with no zeros.
Then there exists a function $f : \mathbb{R}^2 \to \mathbb{R}$ s.t. $L_\xi f > 0$ (i.e. $Imm_{span\,\xi}(\mathbb{R}^2, \mathbb{R}) \neq \emptyset$).

Some classes of distr. $\mathcal{H}$ with $\mathcal{H}$-free maps in crit. dim.

We generalized Weiner's Lemma in three different directions.
In the first we do not require ξ to be Hamiltonian:

Lemma (1. 1-distributions in $\mathbb{R}^2$)

Let ξ a vector field on $\mathbb{R}^2$ of finite type (i.e. the set of its separatrices is closed and each leaf is inseparable from just finitely many other leaves) with no zeros.
Then there exists a function $f : \mathbb{R}^2 \to \mathbb{R}$ s.t. $L_\xi f > 0$ (i.e. $Imm_{span\,\xi}(\mathbb{R}^2, \mathbb{R}) \neq \emptyset$).

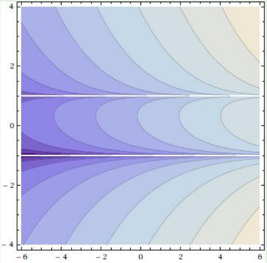
Proposition 2

Under the hypotheses above, $Free_{span\,\xi}(\mathbb{R}^2, \mathbb{R}^2) \neq \emptyset$.

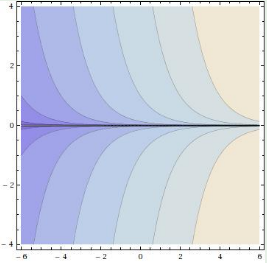
Some classes of distr. $\mathcal{H}$ with $\mathcal{H}$-free maps in crit. dim.

Example

Consider $\xi = (3y-1)\partial_x + (1-y^2)\partial_y$ on $\mathbb{R}^2$. This vector field is not Ham. w/resp. to any symplectic structure on $\mathbb{R}^2$ but it is transversal to the level sets of $f(x,y) = ye^x$.
Hence e.g. (ye^x, y^2e^{2x}) is *span* ξ-free.



Int. Traj. of ξ



Level sets of f

Some classes of distr. $\mathcal{H}$ with $\mathcal{H}$-free maps in crit. dim.

In the 2nd we realize that a Hamiltonian system on $\mathbb{R}^2$ is a compl. integr. syst. (CIS) and we consider more general CISs:

Lemma (2. Lagrangian n-distr. on symplectic mfds (M^{2n}, Ω))

Let (M^{2n}, Ω) be a symplectic manifold admitting a CIS $\{I_1, \cdots, I_n\}$, $\mathcal{H} \subset TM$ the n-dimensional Lagrangian distribution $\mathcal{H} = \cap_{i=1}^n \ker dI_i$ and $\mathcal{F}$ the corresponding Lagrangian foliation. Assume that the Hamiltonian vector fields ξ_i associated to the I_i are all complete and that every leaf of $\mathcal{F}$ has no compact component. Then there exist n smooth functions f^i on M s.t. $L_{\xi_i} f^j = 0$, $i \neq j$, and $L_{\xi_i} f^i > 0$ (i.e. $Imm_{\mathcal{H}}(M, \mathbb{R}^n) \neq \emptyset$).

Some classes of distr. $\mathcal{H}$ with $\mathcal{H}$-free maps in crit. dim.

In the 2nd we realize that a Hamiltonian system on $\mathbb{R}^2$ is a compl. integr. syst. (CIS) and we consider more general CISs:

Lemma (2. Lagrangian n-distr. on symplectic mfds (M^{2n}, Ω))

Let (M^{2n}, Ω) be a symplectic manifold admitting a CIS $\{I_1, \cdots, I_n\}$, $\mathcal{H} \subset TM$ the n-dimensional Lagrangian distribution $\mathcal{H} = \cap_{i=1}^n \ker dI_i$ and $\mathcal{F}$ the corresponding Lagrangian foliation. Assume that the Hamiltonian vector fields ξ_i associated to the I_i are all complete and that every leaf of $\mathcal{F}$ has no compact component. Then there exist n smooth functions f^i on M s.t. $L_{\xi_i} f^j = 0$, $i \neq j$, and $L_{\xi_i} f^i > 0$ (i.e. $Imm_{\mathcal{H}}(M, \mathbb{R}^n) \neq \emptyset$).

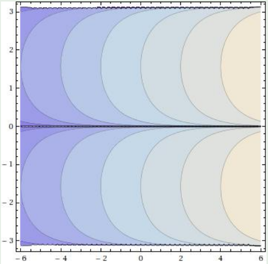
Proposition 3

Under the hypotheses above, $Free_{\mathcal{H}}(M, \mathbb{R}^{q_n}) \neq \emptyset$.

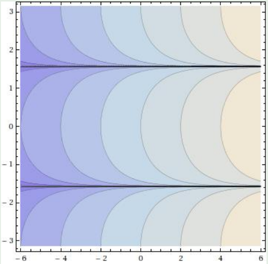
Some classes of distr. $\mathcal{H}$ with $\mathcal{H}$-free maps in crit. dim.

Example

Consider the CIS $\{I\}$ on $\mathbb{R} \times \mathbb{S}^1$ with $I(z, \phi) = \sin\phi e^z$. The corresponding Hamiltonian vector field w/resp to $\Omega = dz \wedge d\phi$ is $\xi = e^z(\cos\phi\partial_z - \sin\phi\partial_\phi)$. The level sets of I are non-compact so we know there exists f s.t. $L_\xi f > 0$. E.g. $L_\xi(\cos\phi e^z) = e^{2z} > 0$. Hence the map $(\cos\phi e^z, \cos^2\phi e^{2z})$ is *span* ξ-free.



Int. Traj. of ξ



Level sets of f

Some classes of distr. $\mathcal{H}$ with $\mathcal{H}$-free maps in crit. dim.

In the 3rd we think a Ham. system on $\mathbb{R}^2$ as a Poisson-Riemann system (PRS) and consider PRSs of higher dimension:

Theorem (3. Riemann-Poisson bracket)

Let M be an n-dim. oriented Riemannian mfd, $H = \{h_1, \cdots, h_{n-2}\}$ a set of $n-2$ functions funct. ind. at every point and $\{f, g\}_H = *[dh_1 \wedge \cdots \wedge dh_{n-2} \wedge df \wedge dg]$ the corresponding Riemann-Poisson bracket. Then, if $h \in C^\infty(M)$ is funct. ind. from all the h_i and $\mathcal{H} = span\{\xi_h\}$ for $\xi_h(f) = \{h, f\}_H$, there exists a (possibly multivalued) smooth function $F : M \to \mathbb{R}$ such that $L_{\xi_h} f > 0$ (i.e. $Imm_{\mathcal{H}}(M, \mathbb{R}) \neq \emptyset$).

Some classes of distr. $\mathcal{H}$ with $\mathcal{H}$-free maps in crit. dim.

In the 3rd we think a Ham. system on $\mathbb{R}^2$ as a Poisson-Riemann system (PRS) and consider PRSs of higher dimension:

Theorem (3. Riemann-Poisson bracket)

Let M be an n-dim. oriented Riemannian mfd, $H = \{h_1, \cdots, h_{n-2}\}$ a set of $n-2$ functions funct. ind. at every point and $\{f, g\}_H = *[dh_1 \wedge \cdots \wedge dh_{n-2} \wedge df \wedge dg]$ the corresponding Riemann-Poisson bracket. Then, if $h \in C^\infty(M)$ is funct. ind. from all the h_i and $\mathcal{H} = span\{\xi_h\}$ for $\xi_h(f) = \{h, f\}_H$, there exists a (possibly multivalued) smooth function $F : M \to \mathbb{R}$ such that $L_{\xi_h} f > 0$ (i.e. $Imm_{\mathcal{H}}(M, \mathbb{R}) \neq \emptyset$).

Proposition 4

Under the hyp. above and if f is single-val., $Free_{\mathcal{H}}(M, \mathbb{R}^2) \neq \emptyset$.

Open Problems

1. h-principle for $\mathcal{H}$-immersions and $\mathcal{H}$-free maps: what can be said in general for the existence of $\mathcal{H}$-immersions and $\mathcal{H}$-free maps in the intermediate range? (i.e. when q is not big enough for genericity but not small enough to rule out their existence)

Open Problems

1. h-principle for $\mathcal{H}$-immersions and $\mathcal{H}$-free maps: what can be said in general for the existence of $\mathcal{H}$-immersions and $\mathcal{H}$-free maps in the intermediate range? (i.e. when q is not big enough for genericity but not small enough to rule out their existence)
2. Find explicit $\mathcal{H}$-free maps in critical dimension for more classes of distributions.

Open Problems

1. h-principle for $\mathcal{H}$-immersions and $\mathcal{H}$-free maps: what can be said in general for the existence of $\mathcal{H}$-immersions and $\mathcal{H}$-free maps in the intermediate range? (i.e. when q is not big enough for genericity but not small enough to rule out their existence)
2. Find explicit $\mathcal{H}$-free maps in critical dimension for more classes of distributions.
3. Prove (or disprove!) the "main goal", i.e. that for every $\mathcal{H}$ the operator $\mathcal{D}_{\mathcal{H},q}$ is surjective for q large enough.

Open Problems

1. h-principle for $\mathcal{H}$-immersions and $\mathcal{H}$-free maps: what can be said in general for the existence of $\mathcal{H}$-immersions and $\mathcal{H}$-free maps in the intermediate range? (i.e. when q is not big enough for genericity but not small enough to rule out their existence)
2. Find explicit $\mathcal{H}$-free maps in critical dimension for more classes of distributions.
3. Prove (or disprove!) the "main goal", i.e. that for every $\mathcal{H}$ the operator $\mathcal{D}_{\mathcal{H},q}$ is surjective for q large enough.
4. Study what happens for other structures on $\mathcal{H}$ (symplectic, contact, connections and so on).

Bibliography

1. M. Gromov, *Partial Differential Relations*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 9, Springer-Verlag
2. M. Gromov and V.A. Rokhlin, *Immersions and embeddings in Riemannian geometry*, RMS **25** (1970), 1-57
3. J. Weiner, *First integrals for a direction field on a simply connected plane domain*, Pac. J. of Math. **132**:1 (1988), 195-208
4. G. D'Ambra, A. Loi, -, *Partially Isometric Immersions and Free Maps*, to appear on Geom. Dedicata, http://arxiv.org/abs/1007.3024
5. -, *Solvability of the cohomological equation for regular vector fields on the plane*, http://arxiv.org/abs/1007.3016
6. T. Gramchev, A. Kirilov, -, *Global Solvability in Functional Spaces for Smooth Nonsingular Vector Fields in the Plane*, http://arxiv.org/abs/1001.2121