Dynamics of the Logistic Map

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joint work with J.A. Yorke (UMD)

Plan of the presentation:

- Fundamental concepts

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- General structure of a Dynamical System

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- The case of the Logistic Map

Fundamental Concepts

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Example: given a complete vector field v on a smooth manifold M, the solutions of the ODE

$$\dot{x}(t) = v(x(t))$$

generate a flow $\Phi : \mathbb{R} \times M \to M$ so that $x(t) = \Phi_t(x_0)$ is the solution to the ODE above with initial conditions $x(0) = x_0$.

Discrete Dynamical Systems

When $T = \mathbb{Z}$, a flow is simply given by the iteration of a continuous map $f : M \to M$, namely

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From now on we wil consider only discrete dynamical systems.

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Example: every point in a finite orbit $x, f(x), f^2(x), \dots, f^{n-1}(x)$, with $f^n(x) = x$, is a non-wandering point.

Attractors and Repellors

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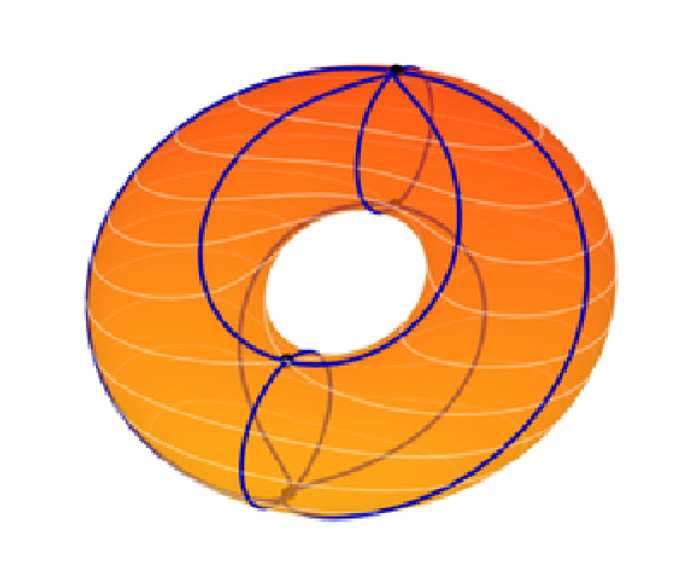
In the rest of the talk we will review the main general results in this direction and will show that even in the most elementary case of the logistic map the situation can be far from trivial.

Fundamental Theorem of Dynamics

Morse decomposition (20s)



Morse-Smale functions (60s)

In Sixties, S. Smale vastly generalized this observation by Morse to *Axiom-A* diffeomorphisms, namely diffeomorphisms f for which Ω_f is hyperbolic (the tangent space decomposes into attracting and repelling subspaces) and periodic points are dense in Ω_f. The celebrated Smale's horseshoe map is an Axiom-A map.

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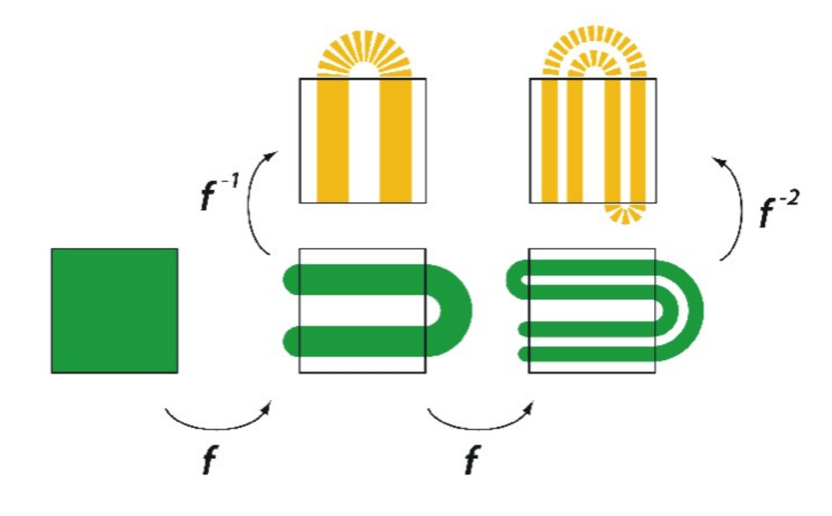
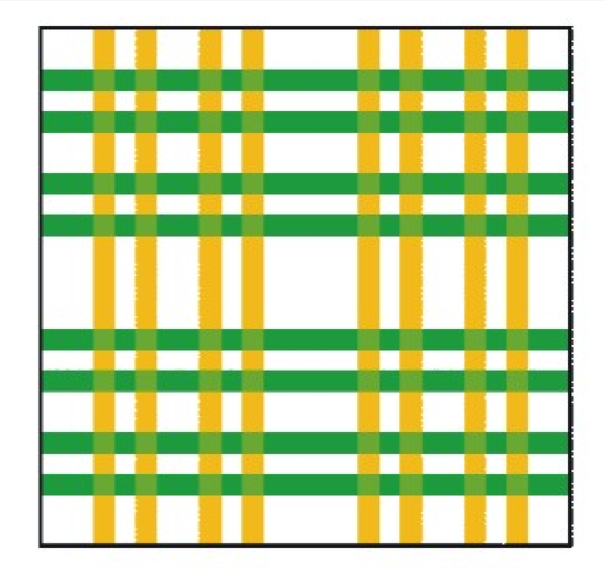
More technically, there exists a Lyapunov function L such that, for every x outside of Ω_f, $L(f(x)) < L(x)$ and L assumes different constant values on every invariant component of Ω_f.

Smale's horseshoe map



Horseshoe's attractor



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This is the set of all points $\mathcal{R}_{\{}$ with the following property: for every $\varepsilon > 0$ there is a sequence of points $x_0, x_1, \ldots, x_n$ such that $d(f(x_i), x_{i+1}) < \varepsilon$ and $d(f(x_n), x_0) < \varepsilon$.

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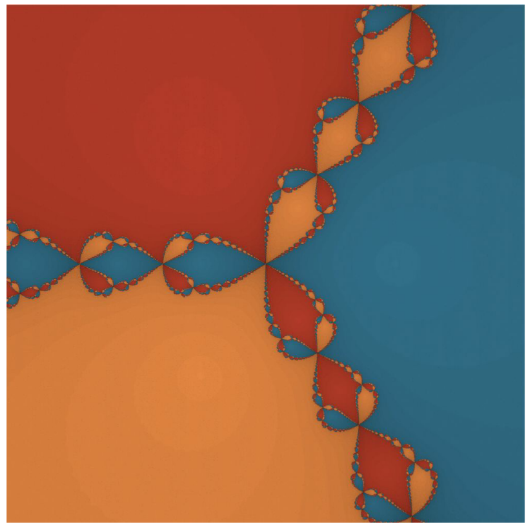
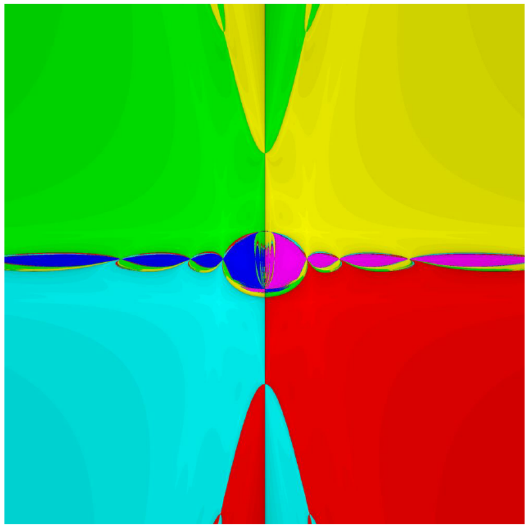
Note that $\Omega_f \subset \mathcal{R}_{\{}$.

Example: Newton map of a complex polynomial



Example: Newton map of a real polynomial map $\mathbb{R}^2 \to \mathbb{R}^2$



The Logistic Map

Some History

The history of the Logistic map

$$f_\mu = \mu x(1-x) : [0,1] \to [0,1], \;\; \mu \in [0,4],$$

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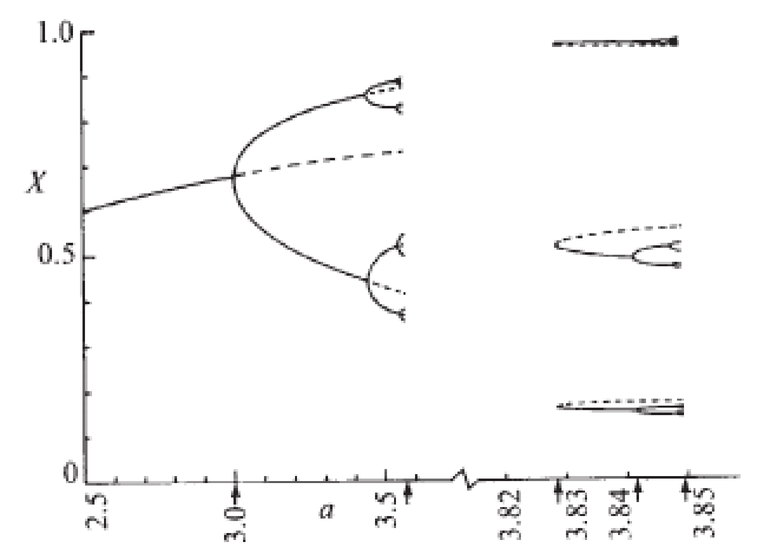
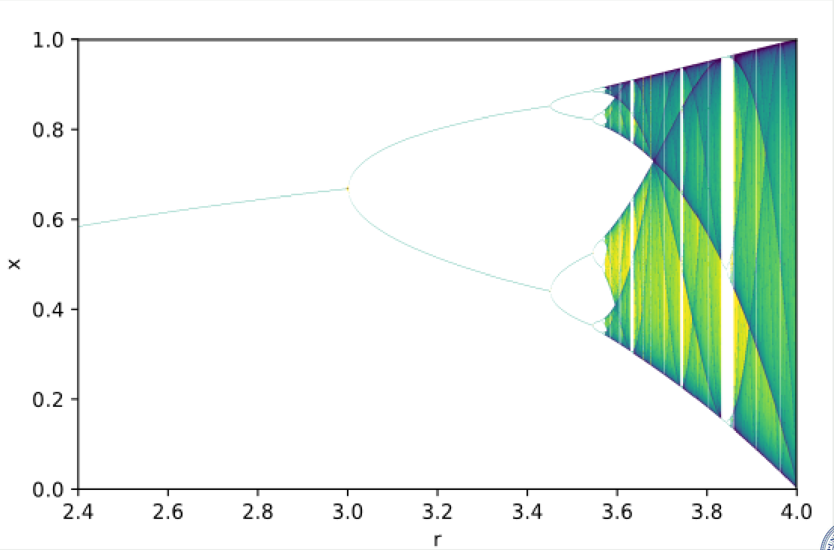
Usually the beginning of the massive interest and study of this map is set to the celebrated paper by biologist R. May “Simple mathematical models with very complicated dynamics”, appeared on *Nature* in 1976.

May's bifurcation diagram (1976)

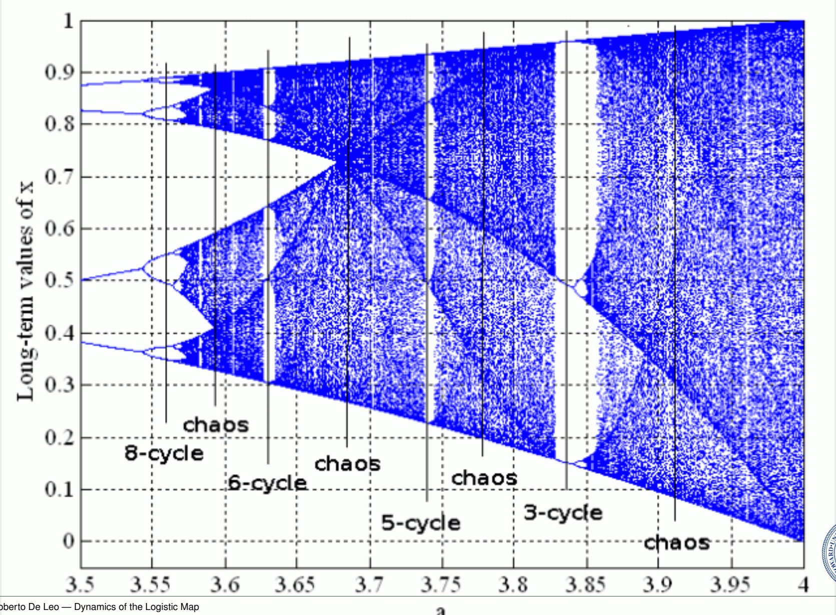


Full bifurcation diagram (1982?)



Myrberg-Feigenbaum point $\mu_{MF} \simeq 3.5699$.

Zoom of the bifurcation diagram



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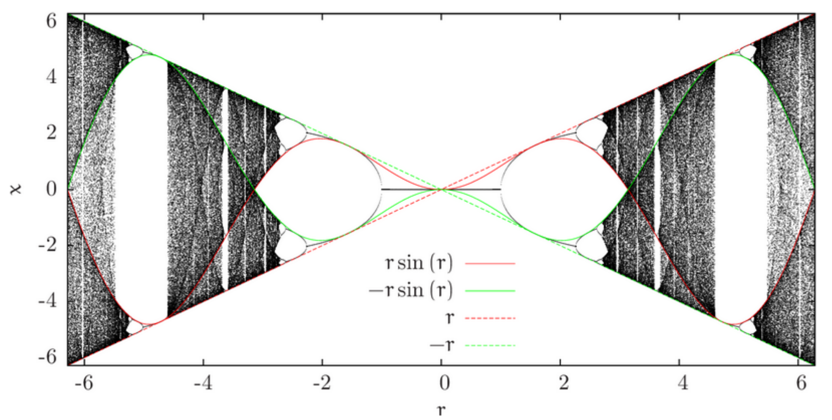
3. how general is this behavior?

Universality of the diagram



In general, this qualitative behavior is shared by all S-unimodal maps, namely maps with negative negative Schwartzian derivative: $Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2$.

S-unimodal maps

A first universal property of S-unimodal maps was discovered numerically by Feigenbaum: given the sequence of bifurcating values μ_n at the left of μ_{MF}, the limit

$$\lim_{n\to\infty} \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n} \simeq 4.6692$$

does not depend on the particular form of f_μ as long as it is S-unimodal!

General properties of S-unimodal maps

Let $f : [0,1] \to [0,1]$ be any S-unimodal map. Define its topological entropy as $\limsup \log \#\{(f^n)'(x) = 0\}$, namely a measure of how fast the number of maxima and minima of f^n is increasing with n. Then:

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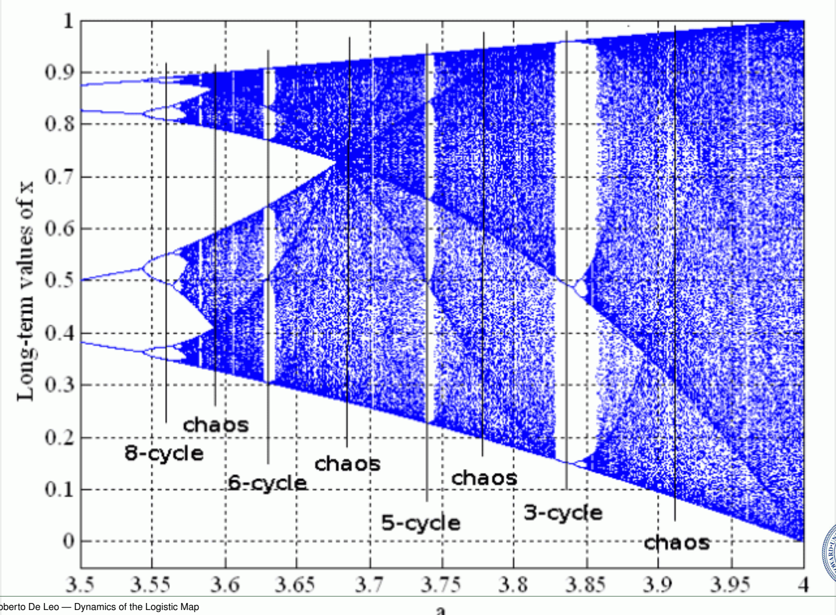
3. when $h(f)=0$, the attractor can be either a cycle of 2^k points or a zero-measure Cantor set and there is no chaos (80s);
4. when $h(f)>0$, the attractor can be either a cycle of $2^k\cdot N$ points or a cycle of $2^k\cdot N$ *intervals* or a Cantor set and the dynamics is chaotic *but only in case of intervals the chaotic dynamics is supported on a set of non-zero measure.*

What is happening for $\mu \geq \mu_{MF}$?

The answer to this question was made rigorous only in 2000s!
The parameter space $[0, 4]$ can be decomposed into 3 disjoint sets R (regular ones, for which A_{f_mu} is a cycle of points), F (non-regular ones, for which A_{f_mu} is a cycle of intervals) and I (those for which the attractor is a Cantor set).
Then R is an open dense set (Graczyk and Swiatek, 1997), F is a Cantor set of non-zero measure (Jakobson, 1981) and I is a Cantor set of zero measure (Lyubich, 2002).

Zoom of the bifurcation diagram



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To date, though, there is no work in literature that considers the *dynamical* structure of Ω_f, namely the dynamics of f close to the components of Ω_f. Notice that this question is equivalent to applying the Moser-Smale-Conley paradigm to the logistic map.

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As a corollary of a result mentioned earlier, arbitrarily close to every parameter μ giving rise to chaotic dynamics there is a parameter with an infinite tower. We claim that this property is general among discrete dynamical systems in any dimension (work in progress).