Solvability of the Cohomological Equation for regular vector fields on the plane

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Outline

Broad Statement of the Problem

Ingredients:

M smooth manifold with coordinates (x^α), $\alpha = 1, \cdots, m$

$\xi = (\xi^\alpha)$ smooth vector field on M

$\Phi^t_\xi : M \to M$ flow of ξ, i.e. $\xi_p = \frac{d}{dt}\Phi^t_\xi(p)\Big|_{t=0}$, $\forall p \in M$

ξ can be seen as a 1st-order linear Partial Diff. Op. on $C^\infty(M)$

$$L_\xi f(p) := \frac{d}{dt} f(\Phi^t_\xi(p))\Big|_{t=0} = \xi^\alpha \frac{\partial f}{\partial x^\alpha}(p)$$

Natural Questions:

1. How to characterize the sets $L_\xi(C^r(M)) \cap C^k(M)$, $L_\xi(W^{l,p}_{loc}(M)) \cap C^k(M)$?
2. How do these sets depend on the *topology* and *geometry* of the foliation $\mathcal{F}_\xi$ of all integral trajectories of ξ?

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This problem is equivalent to characterizing the functions g for which it is solvable the so-called Cohomological Equation

$$L_\xi f = g\,, \quad g \in C^k(M)$$

It is well-known that the solvability of the CE is locally trivial since, by the method of characteristics, if τ is some curve everywhere transversal to ξ's flow, then

$$f(p) = f_\tau(p_0) + \int_0^{t_{p,p_0}} g\left(\Phi^t_\xi(p)\right)\, dt$$

where f_τ is any function defined on τ, p_0 the (unique) point of τ s.t. p_0 and p belong to the same leaf of $\mathcal{F}_\xi$ and t_{p,p_0} is the time needed to travel between these two points under Φ^t_ξ.

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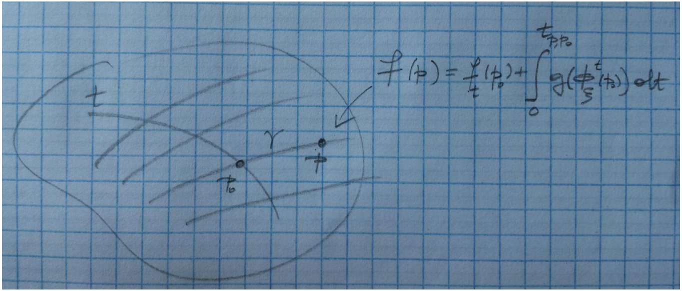

Cohomological Equation

This f provides a global solution iff γ covers M under Φ^t_ξ:

Theorem (Duistermaat & Hormander, 1972)

Let M be an open manifold. Then $L_\xi(C^\infty(M)) = C^\infty(M)$ iff ξ admits a global transversal, i.e. iff $\mathcal{F}_\xi \simeq \mathbb{R}$.

The global solvability of the CE was recently investigated for compact surfaces by S.P. Novikov in case of smooth functions:

S.P. Novikov, "*Dynamical Systems and Differential Forms. Low Dimensional Hamiltonian Systems*", arXiv:math/0701461v3

and by G. Forni in case of Sobolev spaces of weakly diff. functions which are zero in some nbhd of the zeros of ξ:

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Cohomological Equation in $\mathbb{R}^2$

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General Setting

CE in $\mathbb{R}^2$

A Basic Example

General Results & Open Problems

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Theorem (Novikov, 2007)

Let (M, Ω) be a compact symplectic surface of genus ≥ 2. Then, for a generic $\xi \in Ham_\Omega(M)$, the operator $L_\xi : C^\infty(M) \to C^\infty(M)$ has an infinite-dimensional cokernel, namely there infinitely many linearly independent functions not in the image of L_ξ.

Theorem (Forni, 1997)

Let (M, Ω) be a compact symplectic surface of genus ≥ 2 and $\Sigma \subset M$ a finite set. Then there is a $r \geq 1$ such that, for almost all $\xi \in Ham_\Omega(M)$ s.t. $\{\xi = 0\} = \Sigma$, if $g \in W^{r-1,2}(M)$ has compact support in $M \setminus \Sigma$ and $\int_M g\,\Omega = 0$ then $L_\xi f = g$ has a distributional solution in $W^{-r,2}_{loc}(M \setminus \Sigma)$.

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Specific Statement of the Problem

We are interested in the particular case $M = \mathbb{R}^2$ when ξ is a vector field without zeros.

Definition

If ξ has no zero we call it a *regular* vector field. Analog. we call a smooth function f *regular* if its differential df is never zero.

Even with these very strong restriction, the problem is still rich and non-trivial.

Definition

The set of integral trajectories of ξ foliates the plane as the disjoint union of such trajectories. We will use $\mathcal{F}_\xi$ to indicate this foliation.

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General Motivations

- this is the simplest PDE on manifolds;
- ξ and $(g+1)\xi$ have smoothly conjugate flows iff $g \in L_\xi(C^\infty(M))$ [Katok];
- a C^∞ metric $g\,(dt)^2$ on the leaves of $\mathcal{F}_\xi$ arises as the pull-back smooth function $f : M \to \mathbb{R}$ iff $g \in L_\xi(C^\infty(M))$ [A. Loi, G. D'Ambra & RdL];

Motivations Specific to the plane

- ξ is a regular Hamiltonian vector field for some symplectic structure [i.e. there exists a volume form Ω on the plane s.t. $L_\xi\Omega = 0$] iff $\ker L_\xi$ contains regular functions, i.e. iff $\mathcal{F}_\xi$'s leaves are the level sets of a regular function.
- there are "counterintuitive" ξ whose characteristics are arranged in such way that $L_\xi f = 0$ has no non-trivial solutions [Wazewsky].

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References on Regular Foliations on $\mathbb{R}^2$

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- J.L. Weiner, **First integrals for a direction field on a simply connected plane domain**, Pac. J. of Math., 132:1, **1988**, 195-208
- L. Mazzi and M. Sabatini, **On the existence of global first integrals in the plane**, Ann. di Mat. Pura ed Appl., IV, **1993**, 143-160

Fundamental Results

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A function f is a First Integral for ξ if $L_\xi f = 0$ and f is regular. It is a *weak* FI if $df = 0$ is a cod-1 submfd. It is a C^0 FI if it is not constant on any open set.

Theorem (Wazewski 1934)

There exists a $\xi \in \mathfrak{X}_r(\mathbb{R}^2)$ with no C^1 first integral.

Theorem (Kamke 1936)

The restriction of any $\xi \in \mathfrak{X}_r(\mathbb{R}^2)$ to a bounded open domain has a C^∞ first integral.

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Solvability and Smooth Structure of leaf space

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The leaf space $\mathcal{F}_\xi$ of $\xi \in \mathfrak{X}_r(\mathbb{R}^2)$ is a possibly non-Hausdorff 2nd-countable simply-connected oriented 1-dimensional smooth manifold and viceversa.

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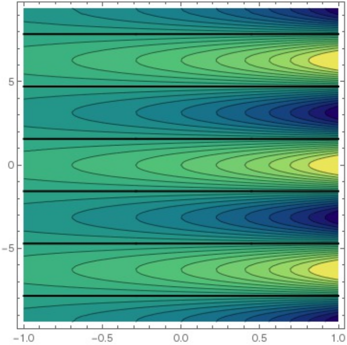
Separatrices of planar vector fields

In this subject it is central the concept of separatrices of a vector field ξ.

Separatrices are characteristics of ξ that cannot be separated, in the quotient topology, from some other integral trajectory.

Clearly $\xi \simeq const$ iff it has separatrices.

E.g. $y = \pm\pi/2, \pm 3\pi/2, \dots$ are separatrices of $\xi = (\sin y, \cos y)$



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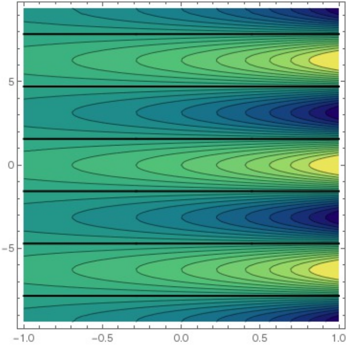

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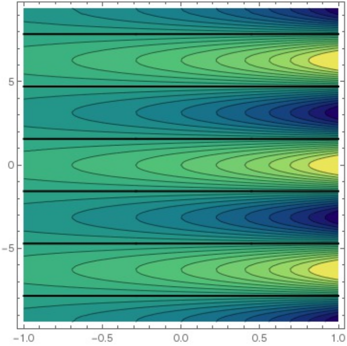

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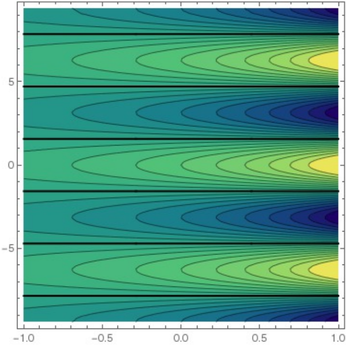
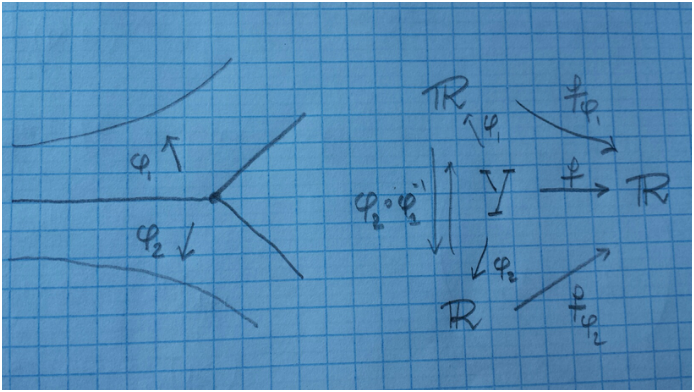


Paradigmatic example: the Y space

The simplest non-Hausdorff 1-dim mfd is the "letter $\mathbf{Y}$",

namely the quotient of the disjoint union of two lines $r_{1,2}$ under the equivalence relation $x \sim y$ iff $x = y$, $x, y < 0$.



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A smooth structure on $\mathbf{Y}$ is given by a pair of charts $\varphi_{1,2} : \mathbb{R} \to r_{1,2}$ s.t. the coordinate changes $\varphi_1^{-1}\varphi_2$ and $\varphi_2^{-1}\varphi_1$ are smooth diffeomorphisms of $(-\infty, 0)$ in itself.

The key point here is that $\varphi_1^{-1}\varphi_2$ (or its inverse) can be divergent at $x = 0$ and it can do that with different speeds.

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Two inequivalent smooth structures on $\mathbf{Y}$:

1. $\mathcal{A}_1 = \{\varphi_1(t) = t, \varphi_2(t) = t\}$. Then $C^\infty(\mathbf{Y}_{\mathcal{A}_1})$ contains regular functions, e.g. $f_{\phi_1}(t) = t = f_{\phi_2}(t)$.
2. $\mathcal{A}_3 = \{\varphi_1(t) = t, \varphi_2(t) = t^3\}$. Then $C^\infty(\mathbf{Y}_{\mathcal{A}_3})$ contains no regular functions!

Indeed $f_2(t) = f_1(\varphi_1\varphi_2^{-1}(t))$ so that, in $\mathcal{A}_3$,

$$f_1'(0) = \left[(\varphi_2\varphi_1^{-1}(t))' f_2'(\varphi_2\varphi_1^{-1}(t))\right]\Big|_{t=0} = [3t^2 f_2'(t^3)]\Big|_{t=0} = 0$$

as long as f_2 is C^1.

Note that we get an inequivalent smooth structure $\mathcal{A}_{2k+1}$ for every $k = 0, 1, \ldots$ by setting $\varphi_2(t) = t^{2k+1}$.

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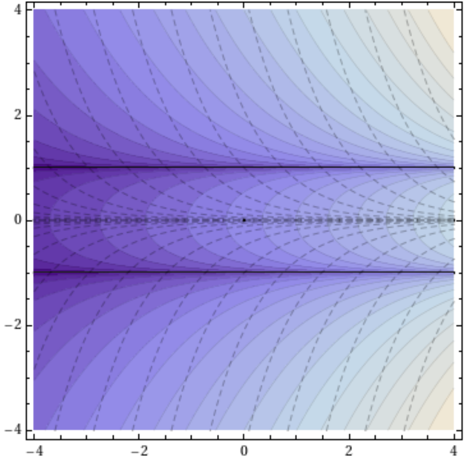
The case $\xi = (2y, 1 - y^2)$

This picture shows the characteristics of $\xi = (2y, 1 - y^2)$ and (dashed) of $\eta = (-2, 2y)$. Note that $\mathcal{F}_\xi \simeq \mathbf{Y}$.



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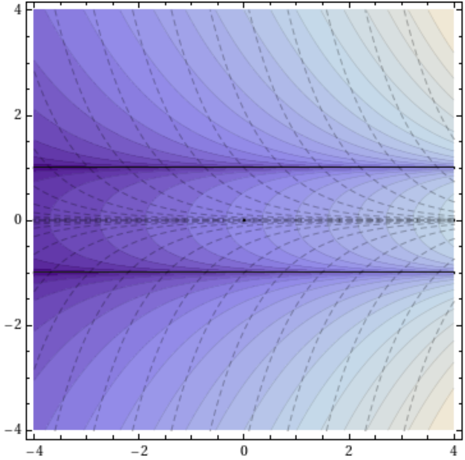

The case $\xi = 2y\partial_x + (1 - y^2)\partial_y$

The two lines $y = \pm 1$ are *inseparable*

in the quotient topology of $\mathcal{F}_\xi$,

in fact they are the only two separatrices of ξ

and are precisely the double point of $\mathbf{Y}$.

The regular vector field η is everywhere transversal to ξ.

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Consider $F(x,y) = (y^2-1)e^x$ and $G(x,y) = 2ye^x$.

By direct calculation it's easy to see that

$$L_\xi F = 0,\ L_\xi G = L_\eta F = 2(1+y^2)e^{2x} > 0,\ L_\eta G = 0.$$

The first and last relations say that both ξ and η are Hamiltonian w/resp to $\Omega = dx \wedge dy$,

the central ones express the mutual trasversality of $\mathcal{F}_\xi$ and $\mathcal{F}_\eta$.

The map $\Phi_{FG} = (F,G) : \mathbb{R}^2 \to \mathbb{R}^2$ *straightens* both foliations:

$$\xi' = \frac{1}{L_\xi G}\xi,\ \eta' = \frac{1}{L_\xi G}\eta$$

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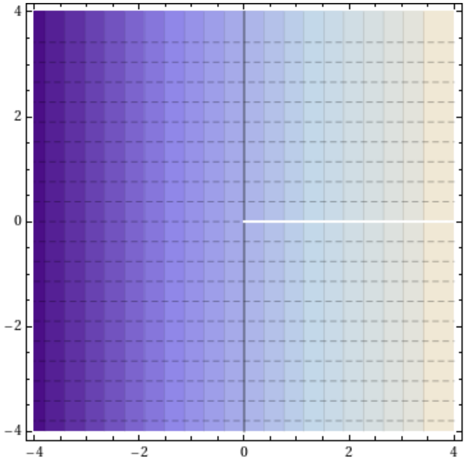
Below is the image $\mathbb{F}_0 \stackrel{\text{def}}{=} \Phi_{FG}(\mathbb{R}^2) = \mathbb{R}^2 \setminus [0,\infty) \times \{0\}$



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In $\mathbb{F}_0 = \mathbb{R}^2 \setminus [0, \infty) \times \{0\}$

the two separatrices are

$\{0\} \times (0, \infty)$ and $\{0\} \times (-\infty, 0)$

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$$L_{\xi'} f = g,\ g \in C^k(\mathbb{R}^2)$$

writes simply as

$$\partial_{y'} f(x', y') = g(x', y'),\ g \in C^k(\mathbb{F}_0).$$

A C^k solution extends "across the two separatrices" to a C^r (resp. $W^{k,p}_{loc}$) function iff the "gap function"

$$gap(x') = \int_{-1}^{1} g(x', y')dy',\ x' < 0,$$

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Cohomological Equation in $\mathbb{R}^2$

R. De Leo

General Setting

CE in $\mathbb{R}^2$

A Basic Example

General Results & Open Problems

For example consider $g(x', y') = y' \in C^\infty(\mathbb{R}^2)$,

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In the original $\mathbb{R}^2$, this means that the solution of

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For every $\xi \in \mathfrak{X}_r(\mathbb{R}^2)$, $\dim\operatorname{coker} L_\xi = \infty$ *iff* ξ *has separatrices.*

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If $\xi \in \mathfrak{X}_r(\mathbb{R}^2)$ *has only isolated separatrices and every separatrix is inseparable from just a finite number of other characteristics,* $L_\xi f > 0$ *has a* C^∞ *solution.*

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Open Problems

1. Does $L_\xi f > 0$ have a C^∞ solution for every $\xi \in \mathfrak{X}_r(\mathbb{R}^2)$?
2. What can be said about $\mathcal{F}_\xi$ for *generic* vector fields in $\xi \in \mathfrak{X}_r(\mathbb{R}^2)$?
3. Study the space of solutions of $\partial_{y'} f = g$ in $\mathbb{F}_0$.
4. Extend these results to the cylinder and other surfaces.
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