Infinitesimal invertibility of the metric inducing operator

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Outline

Three Isometric Embedding Thms of John Nash

Theorem (C^1 embeddings, Nash 1954)

C^0-Riemannian manifolds M^n admit C^1 embeddings into $\mathbb{R}^{2n}$.

Theorem (C^r embeddings, $r > 2$, Nash 1956)

Compact (resp. open) C^r-Riemannian manifolds M^n admit C^r embeddings into $\mathbb{R}^q$, with $q = 3s_n + 4n$ (resp. $q = (n+1)(3s_n + 4n)$) and $s_n = n(n+1)/2$, for every $r = 3, 4, \dots, \infty$.

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The *New Land* of John Nash

Nash, like Columbus, unwillingly discovered a new land. Refining and improving Nash's isometric imbedding results would be like building bigger and faster ships than those in which Columbus had crossed the Atlantic.

But what is this new land? What is its geography, geology, ecology? How can one explore and cultivate this land? What can one build on this land? What is its future?

It may be hard to decide what this land is but it is easy to say what it is not:

> *what Nash discovered is not any part of the Riemannian geometry, neither it has much (if anything at all) to do with the classical PDE.*

The *New Land* of John Nash

Nash's theorems are only superficially similar to the *existence (and non-existence)* results for isometric embeddings that rely on PDE and/or on

relations between intrinsic, i.e. induced Riemannian, and extrinsic geometries of submanifolds in Euclidean spaces.

Nash's results points in the opposite direction:

typically, the geometry of a Riemannian manifold X has no significant influence on its isometric embeddings to $\mathbb{R}^q$.

A Homotopy Perspective in PDEs

Let F, G be two functional spaces and $\mathcal{D} : F \to G$ a Partial Differential Operator (PDO) between them.

Usually the solution of Partial Differential Equations (PDEs) $\mathcal{D}(f) = g$ of interest in analysis (and natural sciences) can be made unique by using appropriate initial or bdary conditions.

Often in Geometry we rather have the opposite situation. The space of solutions is vast and we are rather interested in other questions such as:

1. Assume that $\mathcal{D}(f_0) = g_0$. If g is close enough to g_0, are there solutions to $\mathcal{D}(f) = g$?
2. Consider a C^r family g_λ, $\lambda \in X$. If $\mathcal{D}(f_0) = g_0$, can we find a C^r family f_λ such that $\mathcal{D}(f_\lambda) = g_\lambda$ for all $\lambda \in X$?

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In other words, in this context some more appropriate questions about a PDO $\mathcal{D}: F \to G$ over functional spaces F and G are:

1. what is the maximal domain $F_{op} \subset F$ over which $\mathcal{D}: F_{op} \to G$ is an open map?
2. what is the maximal domain $F_S \subset F$ over which $\mathcal{D}: F_S \to G$ is a Serre fibration?

Remark: throughout this talk all functional spaces will be endowed with the Withney C^∞ topology, so that sets defined through *open conditions* will be *open*.

E.g. with this topology the set of immersions $Imm^\infty(M^n, \mathbb{R}^q)$ is an *open* subset of $C^\infty(M^n, \mathbb{R}^q)$.

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A *Serre fibration* case

1. M^n = smooth n-dimensional manifold
2. $\mathcal{G}^\infty(M^n)$ = smooth Riemannian metrics on M^n
3. $\mathcal{D} : Imm^\infty(M^n, \mathbb{R}^q) \to \mathcal{G}^\infty(M^n)$ is the pull-back map $\mathcal{D}(f) = f^*(euc_q)$.
4. $Free^\infty(M^n, \mathbb{R}^q)$ = smooth maps $f : M^n \to \mathbb{R}^q$ s.t. the $q \times (n + s_n)$ matrix $(\partial_\alpha f^i, \partial_{\alpha\beta} f^i)$ is full rk at every pt.

Theorem (Nash, Gromov)

If M^n is compact, the restr. $\mathcal{D} : Free^\infty(M^n, \mathbb{R}^q) \to \mathcal{G}^\infty(M^n)$ is a Serre fibration for $q \geq s_n + 2n + 3$.

Conjecture (Gromov)

The condition above can be weakened to $q \geq s_n + n + 1$.

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Theorem (Nash, Gromov)

The pull-back map $\mathcal{D} : C^\infty(M^n, \mathbb{R}^q) \to \mathcal{G}^\infty(M^n)$ is open over an open dense subset $\mathcal{F} \subset C^\infty(M^n, \mathbb{R}^q)$ for all $q \geq s_n + 2n$.

Remark: the dense set $\mathcal{F}$ above is precisely $Free^\infty(M^n, \mathbb{R}^q)$. This set is dense in $C^\infty(M^n, \mathbb{R}^q)$ for all $q \geq s_n + 2n$ by transversality arguments and is empty for $q < s_n + n$.

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Main Result

Conjecture (Gromov, 1986 (see also Bull. of AMS, 54:2, 2017))

The pull-back map $\mathcal{D} : C^\infty(M^n, \mathbb{R}^q) \to \mathcal{G}^\infty(M^n)$ is an open map over an open dense (weaker version: non-empty) subset $\mathcal{F} \subset C^\infty(M^n, \mathbb{R}^q)$ for all $q \geq s_n + n - \sqrt{n/2}$.

Theorem (RdL, 2017)

The pull-back map $\mathcal{D} : C^\infty(M^n, \mathbb{R}^q) \to \mathcal{G}^\infty(M^n)$ is an open map over a non-empty open subset $\mathcal{F} \subset C^\infty(M^n, \mathbb{R}^q)$ for all $q \geq s_n + n - \sqrt{n/2} + 1/2$.

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Definition of Partial Differential Operator

1. $F \to E$, $G \to E$ fiber bundles
2. $J^r F \to E$ = bundle of r-jets of sections of $F \to E$
3. $\Gamma^r F = C^r$ sect's of $F \to E$, $\Gamma^0 G = C^0$ sect's of $G \to E$

Definition

By a PDO of order r over F with values in G we mean a map $\mathcal{L}_r : \Gamma^r F \to \Gamma^0 G$ such that $\mathcal{L}_r(f)\big|_x = (j^r_x f)^* \Lambda_r$ for some bundle morphism $\Lambda_r : J^r F \to G$.

In coords (x^α) on E, (x^α, y^i) on F and (x^α, z^a) on G,

$$\mathcal{L}_r(f)\Big|_x = (x^\alpha, \Lambda^a(x^\alpha, \partial_{\alpha_1} f^i|_x, \dots, \partial_{\alpha_1 \dots \alpha_r} f^i|_x))$$

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An example of PDO

$E = M^n$

$F = M^n \times \mathbb{R}^q$

$G = S^0_2(M^n)$ – symmetric $(0,2)$ tensors over M^n

The pull-back operator $\mathcal{D}$ is the 1st order quadratic PDO

$$\mathcal{D} : \Gamma^1(M^n \times \mathbb{R}^q) \simeq C^1(M^n, \mathbb{R}^q) \to \Gamma^0(S^0_2(M^n))$$

given in coords by

$$\mathcal{D}(f) = \delta_{ij}\partial_\alpha f^i \partial_\beta f^j \, dx^\alpha \otimes dx^\beta$$

The corresponding bundle morphism

$$\Delta : J^1(M^n \times \mathbb{R}^q) \simeq J^1(M^n, \mathbb{R}^q) \to S^0_2(M^n)$$

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$$\Delta : J^1(M^n \times \mathbb{R}^q) \simeq J^1(M^n, \mathbb{R}^q) \to S^0_2(M^n)$$

is given in coords by

$$\Delta(x^\alpha, y^i, y^i_\alpha) = \left(x^\alpha, \delta_{ij} y^i_\alpha y^j_\beta\right)$$

An example of PDO

$E = M^n$
$F = M^n \times \mathbb{R}^q$
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The pull-back operator $\mathcal{D}$ is the 1st order quadratic PDO

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Definition of Linearization

VF = vertical tangent vectors of $TF \to TE$
Given a C^r section $f : E \to F$, set $\Gamma^r_f = \Gamma^r(f^*(VF))$.

Definition

The linearization of $\mathcal{L}_r : \Gamma^r F \to \Gamma^0 G$ at $f \in \Gamma^r(F)$ is the *linear* PDO of order r

$$\ell_{r,f} : \Gamma^r_f \to \Gamma^0 G$$

defined by

$$\ell_{r,f}(\eta) = \frac{d}{dt}\mathcal{L}_r(f_t)\Big|_{t=0},$$

where f_t is any curve of sections s.t. $df/dt|_{t=0} = \eta$.

An example of Linearization

In case of the pull-back map, its linearization

$$d_{r,f} : C^1(M^n, \mathbb{R}^q) \to \Gamma^0(S^0_2(M^n))$$

is given in coordinates by

$$d_{r,f}(\eta) = 2\delta_{ij}\ \partial_\alpha f^i\ \partial_\beta \eta^j$$

Definition of Infinitesimal invertibility

Definition

A PDO $\mathcal{L}_r$ is infinitesimally invertible over $\mathcal{A} \subset \Gamma^r F$ if there is a family of linear PDOs of order s

$$\mathcal{E}_f : \Gamma^s(G) \to \Gamma^0_f\,,\ f \in \mathcal{A}$$

such that:

1. $\mathcal{A} \subset \Gamma^d(F)$ for some $d \geq r$;
2. the map $\mathcal{E} : \mathcal{A} \times \Gamma^s(G) \to \Gamma^0(VF)$ is of order d in the first variable and s in the second;
3. $\ell_r(\mathcal{E}_f(g)) = g$ for all $f \in \mathcal{A} \cap \Gamma^{r+d}F$ and $g \in \Gamma^{s+d}G$.

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An example of Infinitesimal invertibility

The pull-back operator $\mathcal{D} : C^1(M^n, \mathbb{R}^q) \to \Gamma^0(S^0_2(M))$ admits an infinitesimal inverse of defect $d = 2$ and order $s = 0$ (i.e. algebraic!) over $\mathcal{A} = Free^2(M^n, \mathbb{R}^q) \subset C^2(M^n, \mathbb{R}^q)$.
Indeed, the $q \times s_n$ system

$$2\delta_{ij}\, \partial_\alpha f^i\, \partial_\beta \eta^j = \gamma_{\alpha\beta},$$

thanks to the obvious $\partial_\beta(\partial_\alpha f^i\, \eta^j) = \partial^2_{\alpha\beta} f^i\, \eta^j + \partial_\alpha f^i\, \partial_\beta \eta^j$, is implied by the $q \times (n + s_n)$ system

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The Implicit Function Theorem

Theorem (Nash, Gromov, 1986)

If a PDO $\mathcal{L}_r : \Gamma^r F \to \Gamma^0 G$ admits an infinitesimal inversion of order s and defect d over $\mathcal{A} \subset \Gamma^d F$, then the restriction of $\mathcal{L}_r$ to $\mathcal{A}^\infty = \mathcal{A} \cap \Gamma^\infty E$ is an open map.

Corollary (Nash, Gromov)

The pull-back operator $\mathcal{D}$ is open over $Free^\infty(M^n, \mathbb{R}^q)$.

Remark: in particular the corollary implies that $\mathcal{D}$ is open over a dense open set when $q \geq s_n + 2n$ and is void when $q < s_n + n$.

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The problem

We want to show that $\mathcal{D}$ is infinitesimally invertible on some non-empty open subset $\mathcal{F}$ of $C^1(M^n, \mathbb{R}^q)$ even when free maps cannot arise ($q < s_n + n$).

Going back to the $q \times s_n$ system

$$2\delta_{ij}\,\partial_\alpha f^i\,\partial_\beta \eta^j = \gamma_{\alpha\beta},$$

its solutions are given by the union of all solutions of

$$\begin{aligned} \delta_{ij}\,\partial_\alpha f_0^i\,\eta^j &= h_\alpha \\ 2\delta_{ij}\,\partial^2_{\alpha\beta} f_0^i\,\eta^j &= \partial_\alpha h_\beta + \partial_\beta h_\alpha - \gamma_{\alpha\beta}, \end{aligned} \tag{2}$$

for all possible 1-forms $h = h_\alpha dx^\alpha$ over M^n.

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Full 2-rank maps

Definition

Let $D^2 f = (\partial_\alpha f^i, \partial^2_{\alpha\beta} f^i)$. We say that an immersion $f \in C^2(M^n, \mathbb{R}^q)$ has full 2-rank if $rk\, D^2 f$ is maximal at every point.

Hence if $q \geq s_n + n$ a full 2-rank map is just a Free map.

When $q < s_n + n$, it is an immersion whose 1st and 2nd derivatives span a q-dimensional space at every point.

In other words, the vectors $(\partial_\alpha f^i, \partial^2_{\alpha\beta} f^i)$ satisfy at every point $m = s_n + n - q$ non-trivial linear relations

$$\lambda^\alpha_a \partial_\alpha f + \sum_{\alpha \leq \beta} \lambda^{\alpha\beta}_a \partial^2_{\alpha\beta} f = 0\,, \quad a = 1, \dots, m,$$

where the coefficients λ can be chosen as polynomials in the entries of $D^2 f$.

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Full 2-rank maps

Hence the linear algebraic system (in the q variables η^j)

$$\begin{aligned} \delta_{ij}\,\partial_\alpha f_0^i\,\eta^j &= h_\alpha \\ 2\delta_{ij}\,\partial^2_{\alpha\beta} f_0^i\,\eta^j &= \partial_\alpha h_\beta + \partial_\beta h_\alpha - \gamma_{\alpha\beta}, \end{aligned} \tag{3}$$

is solvable iff so is the linear PDE system (in the n vars h_α)

$$\lambda^\alpha_a h_\alpha + \sum_{\alpha\le\beta} \lambda^{\alpha\beta}_a (\partial_\alpha h_\beta + \partial_\beta h_\alpha - \gamma_{\alpha\beta}) = 0\,, \quad a = 1,\dots,m$$

(recall that the 1st & 2nd derivatives of f are inside the λ)

Linear Partial Differential Operators

In order to solve the system, we follow Gromov's suggestion to modify his proof about the generic surjectivity of linear PDOs.

Definition

Let $F \to E$ and $G \to E$ be two vector bundles resp. of dimension q and q'. A PDO of order r, $\mathcal{L}_r : \Gamma^r F \to \Gamma^0 G$, is linear if $\mathcal{L}_r(f) = (j^r f)^* \Lambda_r$ for some vector bundle morphism $\Lambda_r : J^r(F) \to G$.

Theorem

If $q > q'$, a generic linear PDO $\mathcal{L}_r : \Gamma^r F \to \Gamma^0 G$ is surjective.

Gromov proof is very general and can be easily adapted to more particular cases, including the PDE in the λ coefficients when f has full 2-rank.

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Algebraic inversion of linear PDE systems

The main idea of Gromov to solve a linear PDE system

$$\mathcal{L}_r(f) = g$$

of q' equations in q variables, $q > q'$, is that a right inverse for $\mathcal{L}_r$ can be found *algebraically*.

Definition

Let $\mathcal{H}$ be an open subset of $J^{r+s}(Hom(J^r(F), G))$.
A $\mathcal{H}$-universal right inverse for a linear PDO of order r

$$\mathcal{L}_r : \Gamma^r F \to \Gamma^0 G,$$

with $\mathcal{L}_r(f) = (j^r f)^* \Lambda_r$, is a PDO

$$\mathcal{M}_s : \Gamma^\infty \mathcal{H} \times \Gamma^\infty G \to \Gamma^\infty F,$$

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The equation $\mathcal{L}_r(\mathcal{M}_s(\mathcal{L}_r, \cdot)) = id$ is a linear PDE system of order r in the coefficients of $\mathcal{M}_s$.

The crucial observation of Gromov is the existence of an idempotent antihomomorphism (formal adjunction)

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$$\mathcal{L}_r^*(g) = \left(\sum_{|A|\le r} \overline{\Lambda}_a^{iA} \partial_A g^a\right) \stackrel{\text{def}}{=} \left(\sum_{|A|\le r} (-1)^{|A|}\partial_A \left[\left(\overline{\Lambda}_a^i\right)^A g^a\right]\right)$$

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The latter, though, is not anymore a PDO but rather an algebraic system and therefore one does not need anything more complex than transversality and combinatorial arguments to prove its solvability.

Remark: counterintuitively, the solutions of $\mathcal{L}_r(f) = g$ obtained this ways are written in terms of the derivatives of the coefficients of $\mathcal{L}_r$ rather than of their integrals!

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Remark: counterintuitively, the solutions of $\mathcal{L}_r(f) = g$ obtained this ways are written in terms of the derivatives of the coefficients of $\mathcal{L}_r$ rather than of their integrals!

A linear 1st order Ordinary Differential Operator

Consider the case $E = \mathbb{R}$, $n = 1$, $q = 2$, $q' = 1$
and a general linear PDO of order 1, namely here

$$F = \mathbb{R} \times \mathbb{R}^2\,,\ G = \mathbb{R} \times \mathbb{R}\,,\ J^1F = \mathbb{R} \times T\mathbb{R}^2$$

and

$$\mathcal{L} : \Gamma^1 F \simeq C^1(\mathbb{R}, \mathbb{R}^2) \to \Gamma^0 G \simeq C^0(\mathbb{R})$$

is defined by

$$\mathcal{L}(x(t), y(t)) = a(t)x(t) + b(t)y(t) + c(t)x'(t) + d(t)y'(t).$$

Then the equation $\mathcal{L}(x(t), y(t)) = g(t)$

is the underdetermined linear 1st order ODE in 2 vars

$$ax + by + cx' + dy' = g$$

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A right inverse of $\mathcal{L}$ of order 0

The the morphism

$$\Lambda : J^1F \simeq \mathbb{R} \times T\mathbb{R}^2 \to G \simeq \mathbb{R} \times \mathbb{R}$$

associated to $\mathcal{L}$ is defined by

$$\Lambda(t, x, y, v_x, v_y) = (t, ax + by + cv_x + dv_y).$$

Consider now the open subset $\mathcal{H} \subset J^1(Hom(J^1F, G)) = J^1(\mathbb{R} \times T^*\mathbb{R}^2)$ of all morphisms Λ such that $ad - bc + cd' - c'd \neq 0$ for all t.

Then there exist a $\mathcal{H}$-universal right inverse of order 0

$$\mathcal{M}_0 g = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} g,$$

namely every operator $\mathcal{L}$ such that $j^1\Lambda(\mathbb{R}) \subset \mathcal{H}$ is surjective.

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Infinitesimal invertibility

R. De Leo

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Finding the coefficients of the 0-order inverse

Indeed

$$\mathcal{L}^*(g) = \begin{pmatrix} a \\ b \end{pmatrix} g - \frac{d}{dt}\left[\begin{pmatrix} c \\ d \end{pmatrix} g\right] = \begin{pmatrix} ag - cg' - c'g \\ bg - dg' - d'g \end{pmatrix}$$

and we look for a 0-order left inverse

$\mathcal{M}_0^* \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = (u(t) \;\; v(t)) \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ of $\mathcal{L}^*$, namely

$$(u \;\; v) \begin{pmatrix} ag - cg' - c'g \\ bg - dg' - d'g \end{pmatrix} = (ua - uc' + vb - vd')g - (uc + vd)g' = g\,.$$

Hence we need to find two functions $u(t)$ and $v(t)$ so that

$$ua - uc' + vb - vd' = 1$$
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A $\mathcal{H}$-universal inverse

The (formal) unique solution of this system

$$u = \frac{d}{ad - bc + cd' - c'd}\,,\quad v = -\frac{c}{ad - bc + cd' - c'd}$$

is well-defined exactly for all $\mathcal{L}$ s.t. $j^1\Lambda(\mathbb{R}) \subset \mathcal{H}$.

The solution provided by this method to

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$$x(t) = \frac{dg}{ad - bc + cd' - c'd}\,,\quad y(t) = -\frac{cg}{ad - bc + cd' - c'd}$$

Remark: the complement of $\mathcal{H}$ has codimension 1 and so $j^1\Lambda(\mathbb{R}) \not\subset \mathcal{H}$ for a generic Λ. Hence we just proved the invertibility of an open non-empty set of linear 1st order differential operators $C^1(\mathbb{R}, \mathbb{R}^2) \to C^0(\mathbb{R})$.

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Looking for a dense $\mathcal{H}$

To find an open dense $\mathcal{H}$ with a $\mathcal{H}$-universal inverse we must consider higher order operators, starting with of order 1:

$$\mathcal{M}_1^* \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} u(t) & v(t) \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} z(t) & w(t) \end{pmatrix} \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

This time, $\mathcal{M}_1^* \mathcal{L}^* g$ has a term in g, one in g' and one in g'',

so $\mathcal{M}_1^* \mathcal{L}^* g = g$ gives 3 polynomial homogeneous equations (with coefficients in $j^2\Lambda$) in the 4 components of $\mathcal{M}_1^*$.

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This way one can find two independent solutions,

respectively with polynomial denominators p and q in the coordinates of the fibers of $J^2(\mathbb{R} \times T^*\mathbb{R}^2)$

and define the two subbundles
$\mathcal{H}_1 = \{p \neq 0\}$ and $\mathcal{H}_2 = \{q \neq 0\}$
so that sections Λ s.t. $j^2\Lambda \subset \mathcal{H}_1$ or $j^2\Lambda \subset \mathcal{H}_2$
admit a universal left inverse.

Since left inverses can be "joined" through a partition of unity, we can build $\mathcal{H}$-universal left/right inverses with $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$.

The complement of $\mathcal{H}$ is the set $\{p = 0 \text{ and } q = 0\}$, whose codimension is 2, so $j^2\Lambda(\mathbb{R}) \subset \mathcal{H}$ for generic Λ.

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Thanks!