On the cohomological equation in the plane for regular vector fields

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The first part of the talk will deal with global cohomological equations associated with smooth vector fields in the plane [motivated by recent work of S.P. Novikov on compact surfaces]. In the second part I will outline some investigations [in collaboration with T. Gramchev (U. of Cagliari)] on global solvability issues of some vector fields of the above type which are not surjective in C^∞.

Isaac, London, 13-18 July 2009

Outline

Broad Statement of the Problem

Ingredients:

M smooth manifold with coordinates (x^α), $\alpha = 1, \cdots, m$

$\xi = (\xi^\alpha)$ smooth vector field on M

$\Phi^t_\xi : M \to M$ flow of ξ, i.e. $\xi_p = \frac{d}{dt}\Phi^t_\xi(p)\Big|_{t=0}$, $\forall p \in M$

ξ can be seen as a 1st-order linear Partial Diff. Op. on $C^\infty(M)$

$$L_\xi f(p) := \frac{d}{dt} f(\Phi^t_\xi(p))\Big|_{t=0} = \xi^\alpha \frac{\partial f}{\partial x^\alpha}(p)$$

Question: how to characterize the image of L_ξ? And how does it depend on the topology of its integral trajectories of ξ?

Remark: The same question can be posed after replacing $C^\infty(M)$ with other functional spaces [e.g. by allowing weak derivatives] and ξ with Ψ-differential 1-st order linear operators.

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Cohomological Equation

This problem is equivalent to characterizing the functions g for which it is solvable the so-called Cohomological Equation

$$L_\xi f = g$$

It is well-known that the solvability of the CE is locally trivial since, by the method of characteristics, if γ is some curve everywhere transversal to ξ's flow, then

$$f(p) = F(p_\gamma) + \int_0^{t_{p,\gamma}} g\left(\Phi^t_\xi(p)\right)\, dt$$

where F is any function defined on γ, p_γ the (unique) point of γ s.t. p_γ and p belong to the same leaf of $\mathcal{F}_\xi$ and $t_{p,\gamma}$ is the time needed to travel between these two points under Φ^t_ξ.

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Cohomological Equation

This f provides a global solution iff γ covers M under Φ^t_ξ:

Theorem (Duistermaat & Hormander, 1972)

Let M be an open manifold. Then $L_\xi(C^\infty(M)) = C^\infty(M)$ iff ξ admits a global transversal, i.e. iff $\mathcal{F}_\xi \simeq \mathbb{R}$.

The global solvability of the CE was recently investigated for compact surfaces by S.P. Novikov in case of smooth functions:

S.P. Novikov, "*Dynamical Systems and Differential Forms. Low Dimensional Hamiltonian Systems*", arXiv:math/0701461v3

and by G. Forni in case of Sobolev spaces of weakly diff. functions which are zero in some nbhd of the zeros of ξ:

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Theorem (Forni, 1995)

Let M be a compact surface of genus ≥ 2. Then there exists an integer l s.t., for a generic $\xi \in \mathfrak{X}(M)$ Hamiltonian with respect to some symplectic structure and with set of zeros Σ and for any $s > l$, there exists a finite number of ξ-invariant distributions $D_k \in H^{-s}(M)$ s.t. the image of the operator $L_\xi : A^s = \{h \in H^s(M) | \overline{\text{supp } h} \subset M \setminus \Sigma\} \to H^{s-l}(M)$ coincides with the intersection of the kernels of the D_k.

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Specific Statement of the Problem

We are interested in the particular case $M = \mathbb{R}^2$ when ξ is a vector field without zeros.

Definition

If ξ has no zero we call it a *regular* vector field. Analog. we call a smooth function f *regular* if its differential df is never zero.

Even with these strong restriction, the problem is still rich and non-trivial.

Definition

The set of integral trajectories of ξ foliates the plane as the disjoint union of such trajectories. We will use $\mathcal{F}_\xi$ to indicate this foliation.

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Motivations

General Motivations

- ξ and $(g+1)\xi$ have smoothly conjugate flows iff $g \in L_\xi(C^\infty(M))$ [Katok];
- a metric $g\,(dt)^2$ on the leaves of $\mathcal{F}_\xi$ arises as the pull-back smooth function $f: M \to \mathbb{R}$ iff $g \in L_\xi(C^\infty(M))$ [RDL, A. Loi & G. D'Ambra, preprint];

Motivations Specific to the plane

- ξ is a regular Hamiltonian vector field for some symplectic structure [i.e. there exists a volume form Ω on the plane s.t. $L_\xi \Omega = 0$] iff ker L_ξ contains regular functions, i.e. iff $\mathcal{F}_\xi$'s leaves are the level sets of a regular function.
- investigating this problem, Wazewsky showed that there are linear 1st-order linear PDEs $L_\xi f = 0$ with no non-trivial solutions.

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Fundamental Results

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General Setting

The problem in the plane

Recent Results

Definition

A function f is a 1-st integral for ξ if $L_\xi f = 0$ and f is regular.

Theorem (Wazewski 1934)

There exists a $\xi \in \mathfrak{X}_r(\mathbb{R}^2)$ with no C^1 1st-Integral.

Theorem (Kamke 1936)

Every restriction to some bounded open domain of a regular plane foliation has a C^∞ 1st-Integral.

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Solvability and Smooth Structure of leaf space

Theorem (Hafliger & Reeb, 1957)

The leaf space $\mathcal{F}_\xi$ of $\xi \in \mathfrak{X}_r(\mathbb{R}^2)$ is a possibly non-Hausdorff 2nd-countable simply-connected oriented 1-dimensional smooth manifold and viceversa.

Theorem (Hafliger & Reeb, 1957)

The non-Hausdorff manifolds above admit more than one inequivalent smooth structure and only one of them admits regular functions.

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Paradigmatic example: the Y space

The simplest non-Hausdorff 1-dim mfd is the letter $\mathbf{Y}$.

Think of it as the quotient of the disjoint union of two lines $r_{1,2}$ under the equivalence relation $x \sim y$ iff $x = y$, $x < 0$.

Then a smooth struct. on $\mathbf{Y}$ is given by a pair of charts $\varphi_{1,2} : \mathbb{R} \to r_{1,2}$ s.t. the coordinate changes $\varphi_1^{-1}\varphi_2$ and $\varphi_2^{-1}\varphi_1$ are smooth diffeomorphisms of $(-\infty, 0)$ in itself.

Two inequiv. structures:

- $\varphi_1(t) = t$, $\varphi_2(t) = t$. Here the function $f : \mathbf{Y} \to \mathbb{R}$ defined in coordinates as $f_1(t) := f(\varphi_1(t)) = t$, $f_2(t) := f(\varphi_2(t)) = t$, is regular.
- $\varphi_1(t) = t$, $\varphi_2(t) = t^3$. Here no regular function exists! Indeed $f_2(t) = f_1(\varphi_1^{-1}\varphi_2(t))$ so that

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Generic regular vector fields

[Preprint sooon available on the ArXives]

Theorem (-, 2009)

If ξ is a generic regular vector field on the plane, then the only smooth functions belonging to ker L_ξ *are the constants.*

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For every $\xi \in \mathfrak{X}_r(\mathbb{R}^2)$ the partial differential inequality $L_\xi f > 0$ admits smooth solutions.

Theorem (-, 2009)

If $\xi \in \mathfrak{X}_r(\mathbb{R}^2)$ is generic then L_ξ has an infinite-dimensional cokernel. The same holds if ξ is generic Hamiltonian.

Solvability of $L_\xi f = g$: a worked out example

(jointly with T. Gramchev)

We are investigating concrete conditions for the global solvability of $L_\xi f = g$ when

$$\xi = (p(y), q(y))$$

Here p and q are a generic pair of polynomials [in particular they do not get zero at the same point].

We present here in detail the particular case

$$\xi = (2y, 1 - y^2)$$

The leaves of $\mathcal{F}_\xi$ (i.e. the characteristics of ξ) are given by

$$\frac{dx}{dy} = \frac{2y}{1 - y^2}, \text{ i.e. } x(y) = \ln|1 - y^2| + c.$$

[same topology as $\mathbf{Y}$]

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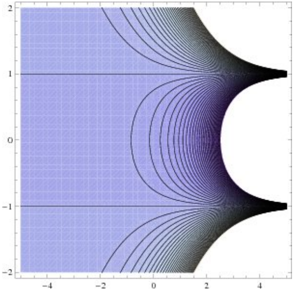
In fact $\mathcal{F}_\xi$ can be seen as the level sets of $F(x,y) = (1-y^2)e^x$



So $L_\xi f = 0$ has non-trivial global solutions but, for example, $L_\xi f = c$, $c \neq 0$, does not. However it has the $L^1_{loc}(\mathbb{R}^2)$ weak (distributional) solution

$$f(x,y) = \frac{c}{2} \ln \left| \frac{1+y}{1-y} \right|$$

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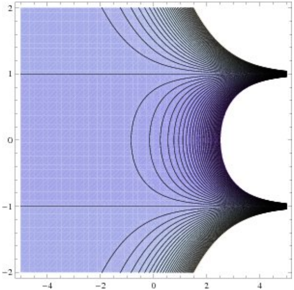

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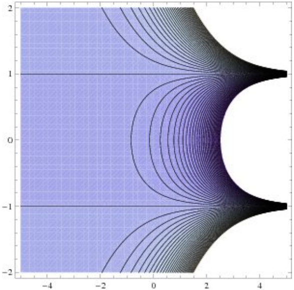

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Solvability of $L_\xi f = g$: a worked out example

More in detail, using the method of characteristics for $L_\xi f = g$, we write formally

$$f(x,y) = \int_0^y \frac{g(x + \frac{1}{2}\ln\left|\frac{1-s^2}{1-y^2}\right|, s)}{1-s^2}\, ds$$

$$= G_+ g(x,y) + G_- g(x,y)$$

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Assume now $g_{\alpha\beta}(x,y) = x^\alpha y^\beta$. Hence

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[applying the multinomial formula]

$$= \pm\sum_{\gamma\in\mathbb{Z}^3_+,|\gamma|=\alpha} \kappa_\gamma x^{\gamma_1}\int_0^y \ln^{\gamma_2}\left|\frac{1-s}{1-y}\right| \ln^{\gamma_3}\left|\frac{1+s}{1+y}\right| s^\beta\, d\left[\ln\left|\frac{1\pm s}{1\pm y}\right|\right]$$

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All terms inside the integral are locally integrable close to $s = \pm 1$, so

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where $\rho_\sigma(x)$, $|\sigma| \leq \alpha+1$, are polynomials.

Since $\left|\ln|z|\right|^k$ near $z = 0$ is L^1 for every $k > 0$, we have shown that f is a well-defined $L^1_{loc}(\mathbb{R}^2)$ function when g is a polynomial.

Similarly, we can show that f is a well-defined $L^1_{loc}(\mathbb{R}^2)$ function when g is subexponential in x.

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On the cohomological equation in the plane for regular vector fields

R. De Leo

General Setting

The problem in the plane

Recent Results

Conclusions

The investigation of the solvability of the linear 1st-order PDE

$$\xi^x f_x + \xi^y f_y = g$$

with ξ regular proved to be a rich and worth studying subject when the topology of the leaves [i.e. integral trajectories] of ξ have a non-trivial topology [so that one cannot apply the Duistermaat-Hormander thm].

Our next move will be finding conditions for its solvability for more functional spaces and consider the case when ξ is a Ψ-differential operator.

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