Asymptotic growth of norms in semigroups of linear automorphisms and Hausdorff dimension of self-projective sets

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Main idea

In a seminal paper D. Sullivan[1] related the Hausdorff dimension $\dim_H R_\Gamma$ of the limit set $R_\Gamma \subset \mathbb{C}P^1$ of a (geometrically finite) Kleinian (i.e. discrete) subgroup $\Gamma \subset PSL_2(\mathbb{C})$ to a critical exponent of Γ defined in the context of hyperbolic geometry.

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In the same spirit here we aim at relating (partly in form of conjecture) an exponent $s_{\boldsymbol{A}}$ associated to some sub*semi*group $\boldsymbol{A}$ of $SL_n^{\pm}(\mathbb{R})$ or $SL_n(\mathbb{C})$ to the Hausdorff or Box dimensions of the limit set $R_{\boldsymbol{A}} \subset KP^{n-1}$, $K = \mathbb{R}$ or $\mathbb{C}$, of the subsemigroup of $PSL_n^{\pm}(\mathbb{R})$ or $PSL_n(\mathbb{C})$ naturally induced by $\boldsymbol{A}$.

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Motivational Example 1

$$\boldsymbol{C}_3 = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \subset SL_3(\mathbb{N})$$

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Let $\Psi_3 \subset Aut(\mathbb{R}P^2)$ be the subsemigroup of projective automorphisms induced by $\boldsymbol{C}_3$
(i.e. $\psi_C([v]) = [C(v)]$ for all $v \in \mathbb{R}^3$, $C \in \boldsymbol{C}_3$)

and T = proj. triangle w/vertices $[1:0:0], [0:1:0], [0:0:1]$.

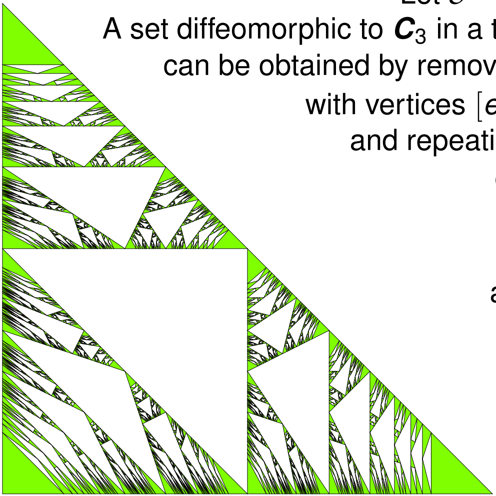
$\Psi_3(T) \subset T$ but some $\psi \in \Psi_3(T)$ is only *parabolic* on T,

i.e. Ψ_3 is a parabolic Iterated Function System (IFS).

There is a unique compact set $\boldsymbol{C}_3 \subset T \subset \mathbb{R}P^2$ invariant by Ψ_3.

Remark: By a conjecture of S.P. Novikov, $1 < \dim_H \boldsymbol{C}_3 < 2$.

The set $\boldsymbol{C}_3$

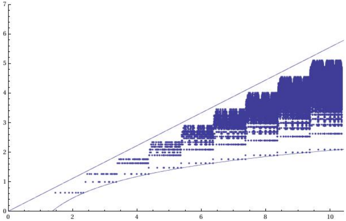


Let $\mathcal{E} = \{e_1, e_2, e_3\}$ a frame of $\mathbb{R}^3$
A set diffeomorphic to $\boldsymbol{C}_3$ in a triangle $T_{\mathcal{E}} = [e_1], [e_2], [e_3]$
can be obtained by removing from $T_{\mathcal{E}}$ the triangle $Z_{\mathcal{E}}$
with vertices $[e_1 + e_2], [e_2 + e_3], [e_3 + e_1]$
and repeating recursively this process
on the remaining triangles.

Left: approximation of $\boldsymbol{C}_3$
at the 7th level of recursion
in the triangle $T_{\mathcal{E}}$ with

$$e_1 = (1, 0, 1),$$
$$e_2 = (0, 1, 1),$$
$$e_3 = (0, 0, 1).$$

Asymptotics of norms of matrices in $\boldsymbol{C}_3$



Left: Log-log plot of norms of matrices in $\boldsymbol{C}_3$ arranged in lexicographic order. The products with fastest growing norm are $\|C_i C_{i+1} C_{i+2} \dots C_{i+k}\| \asymp \phi_3^k$, ϕ_3 Tribonacci constant; those with slowest growing norm are $\|C_i^k\| \asymp k$.

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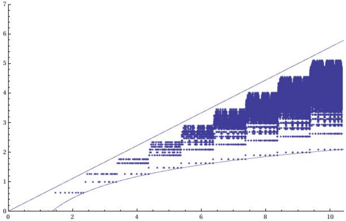

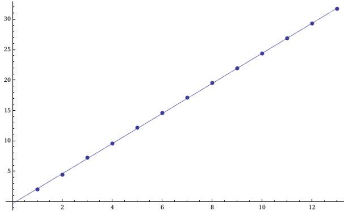

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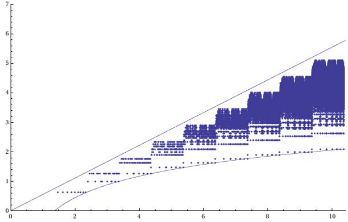
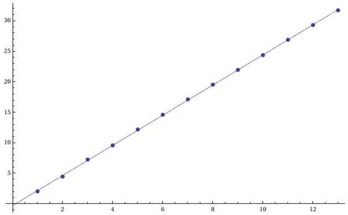
Right: Log-log plot of function $N_{\boldsymbol{C}_3}(r) = \#\{C \in \boldsymbol{C}_3 | \|C\| \leq r\}$.

$$\log N_{\boldsymbol{C}_3}(2^k) \simeq 0.967 + 2.444k$$

Asymptotics of norms of matrices in $\boldsymbol{C}_3$





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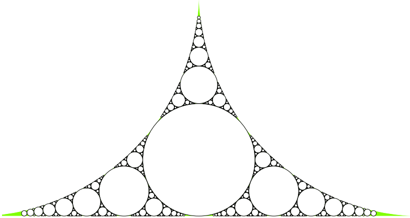
Q: is there a relation between $\dim_H \boldsymbol{C}_3$ and $\lim_{r\to\infty} \frac{\log N_{\boldsymbol{C}_3}(r)}{\log r}$?

Motivational Example 2: the Apollonian gasket A_3



Let $T \subset \mathbb{C}P^1$ the curvilinear triangle above with vertices $[-1:1], [1:1], [i:1]$.

The Apollonian gasket is the unique compact subset of T invariant w/resp. to the semigroup of Klein transformations induced by the semigroup of matrices

$$A_3 = \left\langle \begin{pmatrix} 0 & i \\ i & 2 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \right\rangle \subset SL_2(\mathbb{C})$$

Asymptotics of norms of matrices in $\boldsymbol{A}_3$

In a series of papers dated from 1971 to 1982 D.W. Boyd ultimately proved that $\lim_{r\to\infty} \frac{\log N_{\boldsymbol{H}_3}(r)}{\log r} = \dim_H \boldsymbol{A}_3$, where

$$\boldsymbol{H}_3 = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix} \right\rangle \subset SL_4(\mathbb{N})$$

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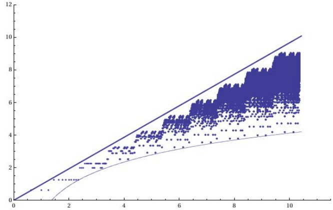
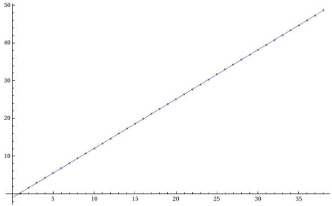
The corresponding log-log plots have the same qualitative behaviour as in the $\boldsymbol{C}_3$ case:





$$\dim_H R_{\boldsymbol{H}} = s_{\boldsymbol{H}} \simeq 1.30568673$$

Asymptotics of norms of matrices in $\boldsymbol{A}_3$

Remark: the proofs relative to the asymptotic growth of norms in a semigroup (our first 3 main results) are based on Boyd's clever arguments introduced for the Apollonian gaskets.

We only claim credit for cleaning them of all particular references to that concrete case and generalizing them in order to apply them to a much more general setting and then for applying this to self-projective sets.

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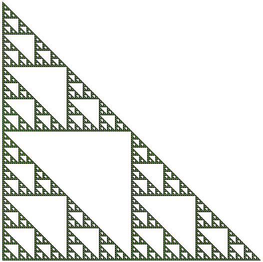
Based on what we shall shortly show, we claim that the Hausdorff dimension of the Apollonian gasket can also be extracted from the semigroup $\boldsymbol{A}_3$:

Conjecture: $$\lim_{r\to\infty} \frac{\log N_{\boldsymbol{A}_3}(r)}{\log r} = 2\dim_H \boldsymbol{A}_3$$

Motivational Example 3: the Sierpinski gasket $\mathbb{S}_3$



Let T be the triangle above. The Sierpinski gasket can be seen, as a *real self-projective fractal*, as the unique compact set in $T \subset \mathbb{R}P^2$ invariant with respect to the semigroup of projective automorphisms induced by

$$\mathbb{S}_3^{\mathbb{R}} = \alpha \left\langle \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \right\rangle \subset SL_3(\mathbb{R}^+).$$

Motivational Example 3: the Sierpinski gasket $\mathbb{S}_3$

Similarly, as a *complex self-projective fractal*, the Sierpinski gasket can be seen as the unique compact set in $T \subset \mathbb{C}P^1$ invariant with respect to the (Kleinian) semigroup of complex projective automorphisms induced by

$$\mathbb{S}_3^{\mathbb{C}} = \alpha \left\langle \begin{pmatrix} 1 & i \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \right\rangle \subset SL_2(\mathbb{C})$$

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In this particular case it is easy to show that

$$\lim_{r\to\infty} \frac{\log N_{\mathbb{S}_3^{\mathbb{R}}}(r)}{\log r} = \frac{3}{2} \dim_H \mathbb{S}_3, \quad \lim_{r\to\infty} \frac{\log N_{\mathbb{S}_3^{\mathbb{C}}}(r)}{\log r} = 2 \dim_H \mathbb{S}_3.$$

Main Problem

Let $S \subset M_n(K)$, $K = \mathbb{R}$ or $\mathbb{C}$, a *finitely generated free* subsemigroup of matrices, $N_S(r) = \#\{A \in S \mid \|A\| \leq r\}$ for any norm $\|\cdot\|$ and Ψ_S the induced semigroup of projective automorphisms of KP^{n-1}.
Let $R_S \subset KP^{n-1}$ be the limit set of Ψ_S.

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Question 2: is there any relation between s and $\dim_H R_S$?

Gasket of matrices

Definition

Denote by $\mathcal{I}^m$ the semigroup of multi-indices $I = i_1 \dots i_k$, $i_j = 1, \dots, m$, with the natural product

$$I \cdot J = i_1 \dots i_k \cdot j_1 \dots j_\ell = i_1 \dots i_k j_1 \dots j_\ell$$

We include in $\mathcal{I}^m$ the neuter element 0, i.e. $0 \cdot I = I \cdot 0 = I$.

A *gasket* is a semigroup homomorphism $\mathcal{A} : \mathcal{I}^m \to M_n(K)$, $\mathcal{A}(0) = \mathbb{1}_n$, such that $N_{\mathcal{A}}(r)$ is finite for every $r \geq 0$.

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We write $A_i = \mathcal{A}(i)$, $A_I = \mathcal{A}(I) = A_{i_1} \cdot A_{i_2} \cdots A_{i_k}$, $k = |I|$.
We say that $\mathcal{A}$ is *generated* by the A_i, $i = 1, \dots m$.

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Remark: when $\mathcal{A}$ is injective then $\mathcal{A}(\mathcal{I}^m) = S \cup \{1\!\!1_n\}$, where S is a free subsemigroup of $M_n(K)$.

Examples of gaskets

Consider $\boldsymbol{C}_2 = \left\langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$. It's easy to show that

$$\alpha k \leq \|A_I\| \leq \beta \phi_2^k,\ k = |I|, \phi_2 = \frac{1+\sqrt{5}}{2}$$

Hence $\boldsymbol{C}_2$ is a gasket. More generally, every free subsemigroup of $M_n(Z)$, $Z = \mathbb{Z}$ or $\mathbb{Z}[i]$, is a gasket since only finitely elements of $M_n(Z)$ lie inside the ball of radius r.

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On the contrary, no homomorphism $\mathcal{A} : \mathcal{I}^2 \to M_n(K)$ with $A_1 = A_2^{-1}$ is a gasket, e.g. because there are infinitely many multi-indices I s.t. $A_I = \mathbb{1}_n$.

Gasket of matrices

Definition

We denote by $\mathcal{J}^m \subset \mathcal{I}^m$ the set of multiindices $J = i_1 i_2 \dots i_2$, with $i_1 \neq i_2$.
We say that a *gasket* is fast if there esists a $c > 0$ s.t.

$$\|A_{IK}\| \geq c\|A_I\|\|A_K\|, \text{ for every } I \in \mathcal{I}^m,\ K \in \mathcal{J}^m \cdot \mathcal{I}^m.$$

We call

$$c_{\boldsymbol{A}} = \inf_{\substack{I \in \mathcal{I}^m \\ K \in \mathcal{J}^m \cdot \mathcal{I}^m}} \frac{\|A_{IK}\|}{\|A_I\|\|A_K\|}$$

the *coefficient* of the gasket.

Examples of fast gaskets

$\boldsymbol{C}_2 = \left\langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$ is fast with coeff. $c = 1/2$ since

$$C_{12l} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ a+2c & b+2d \end{pmatrix}.$$

and so $\|MC_{12l}\| \geq \frac{1}{2}\|M\|\|C_{12l}\|$ for every $M \in SL_2(\mathbb{N})$.

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On the contrary, the gasket generated by $A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $A_2 = A_1$ is not fast since $\|A_1^{k'} A_2 A_1^k\| = 1 + k + k'$.

Zeta Function of a gasket

Definition

The *zeta function* of $\mathcal{A}$ is the series

$$\zeta_{\mathcal{A}}(s) = \sum_{I \in \mathcal{I}^m} \frac{1}{\|A_I\|^s}.$$

We call *exponent* of $\mathcal{A}$ the number $s_{\mathcal{A}}$ defined as follows:

$$s_{\mathcal{A}} = \sup_{s \geq 0} \{\, s \,|\, \zeta_{\mathcal{A}}(s) = \infty \}.$$

Note that, if $s_{\mathcal{A}} < \infty$, we also have that $s_{\mathcal{A}} = \inf_{s \geq 0} \{ s \,|\, \zeta_{\mathcal{A}}(s) < \infty \}$.

Main Result 1

Theorem (Exponent of a fast gasket)

Let $\mathcal{A} : \mathcal{I}^m \to M_n(K)$ be a fast gasket with exponent $s_{\mathcal{A}}$. Then $0 < s_{\mathcal{A}} < \infty$.

Remark

The theorem is constructive in the sense that in the proof are built explicitly two sequences of functions $f_{\mathcal{A},k}(s) \leq g_{\mathcal{A},k}(s)$ such that, for all k from some $\bar{k}$ on, $s_{\mathcal{A}} \in [g^{-1}_{\mathcal{A},k}(1), f^{-1}_{\mathcal{A},k}(1)]$ and $|f^{-1}_{\mathcal{A},k}(1) - g^{-1}_{\mathcal{A},k}(1)| \leq \alpha \log k$ for some $\alpha > 0$.
These sequences can be used effectively to evaluate analytical bounds for the exponents of a few semigroups.

Main Result 2

Theorem (Alternate characterization of a gasket's exponent)

Let $\mathcal{A} : \mathcal{I}^m \to M_n(K)$ be a gasket with $s_{\mathcal{A}} < \infty$. Then the function

$$\xi_{\mathcal{A}}(s) = \lim_{k\to\infty} \left[\sum_{|I|=k} \|A_I\|^{-s} \right]^{\frac{1}{k}}$$

is a well defined log-convex positive strictly decreasing function and satisfies the following properties:

1. $\xi_{\mathcal{A}}(0) = m$;
2. $\xi_{\mathcal{A}}(s) > 1$ *for* $s < s_{\mathcal{A}}$;
3. $\xi_{\mathcal{A}}(s_{\mathcal{A}}) = 1$;
4. $\xi_{\mathcal{A}}(s) < 1$ *for* $s > s_{\mathcal{A}}$ *if* $\mathcal{A}$ *is a hyperbolic gasket;*
5. $\xi_{\mathcal{A}}(s) = 1$ *for* $s > s_{\mathcal{A}}$ *if* $\mathcal{A}$ *is a fast parabolic gasket.*

Main Result 3

Theorem (Exponential growth of norms)

Let $\mathcal{A} : \mathcal{I}^m \to M_n(K)$ be a gasket with $s_\mathcal{A} < \infty$. Then

$$\lim_{k\to\infty} \frac{\log N_\mathcal{A}(k)}{\log k} = s_\mathcal{A}.$$

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Question

Recently Kontorovich and Oh[a] *showed that, in case of the Apollonian gasket, $N(k)$ is* strongly *asymptotic to k^s, namely $N(k) \asymp k^s$ for all $k \in \mathbb{N}$.*
It is natural to ask the following question: is $N_{\mathcal{A}}(k)$ strongly *asymptotic to $k^{s_{\mathcal{A}}}$ for every gasket $\mathcal{A}$ with a finite exponent or is some extra condition necessary?*

[a] A. Kontorovich, H. Oh, "Apollonian circle packings and closed horospheres on hyperbolic 3 manifolds", J. of Am. Math. Soc. 24 (2011)

Main Result 4

Theorem (Residual sets of semigroups of $PSL_2^{\pm}(K)$, $K = \mathbb{R}, \mathbb{C}$)

Let $f_1, \dots, f_m : K^2 \to K^2$ be such that the induced maps $\psi_i \in PSL_2^{\pm}(K)$ satisfy the open set condition with respect to some proper subset $V \subset KP^1$, namely

$$\bigcup_{i=1}^{m} \psi_i(V) \subset V, \quad \psi_i(V) \cap \psi_j(V) = \emptyset, i \neq j$$

Assume that, in some affine chart $\varphi : KP^1 \to K$, the ψ_i are contractions on $\varphi(\overline{V})$ with respect to the Euclidean distance. Let $R_{\mathbf{A}} = \cap_{k=1}^{\infty} \left(\cup_{|I|=k} \psi_I(V)\right)$ be the corresponding residual set. Then $s_{\mathbf{A}} = 2 \dim_H R_{\mathbf{A}}$.

Example

Consider the semigroup F freely generated by

$$F_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, F_2 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}.$$

F induces the subsemigroup $\Psi_F \subset PSL_2(\mathbb{R})$ generated by

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A numerical estimate, obtained by evaluating $N_F(r)$ for $r = 2^k$, $k = 1, \ldots, 28$, gives $s_F \simeq 1.062$, from which the well-known

$$\dim_H R_{\Psi_F} \simeq 0.531$$

(see Falconer, Fractal Geometry, Example 9.8).

Complex projective Sierpinski gaskets

Definition

Let f_1, f_2, f_3 be volume-pres. linear autom. of $\mathbb{C}^2$ with real spectrum and distinct eig. and let $\psi_1, \psi_2, \psi_3 \in PSL_2(\mathbb{C})$ be their corr. proj. automorphisms. Let $[e_i] \in \mathbb{C}P^1$ be a fixed point for ψ_i corresponding to the largest eigenvalue $\lambda \geq 1$ of f_i. We say that the semigroup $\boldsymbol{F}$ gen. by the f_i is a *compl. proj. Sierp. gasket* if:

1. $[f_i(e_j)] = [f_j(e_i)]$ for every i, j with $i \neq j$;
2. the circle Γ_k passing through $[e_i]$, $[e_j]$ (i, j, k is a perm. of $1, 2, 3$) and $[f_i(e_j)]$ is inv. under both f_i and f_j, $i \neq j$;
3. $[f_k(e_i)]$ and $[f_k(e_j)]$ belong to the same connected component of $\mathbb{C}P^1 \setminus \Gamma_k$ for every perm. (i, j, k);
4. the Γ_k do not intersect in the interior of the curvilinear triangle $T_{\boldsymbol{A}} \subset \mathbb{C}P^1$ having as vertices the $[e_i]$ and as sides the segments of the Γ_k with vertices the points $[e_i]$ and $[e_j]$ containing the point $[f_i(e_j)]$.

Complex projective Sierpinski gaskets

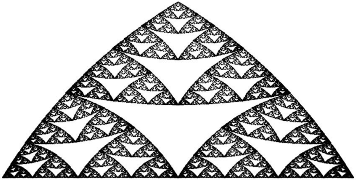
Proposition

Let $f_1, f_2, f_3 : \mathbb{C}^2 \to \mathbb{C}^2$ *be volume-preserving linear automorphisms with real spectrum having resp.* $e_1 = (1,1)$, $e_2 = (i,1)$, $e_3 = (-1,1)$ *as eigenvectors corresp. to the largest module's eigenvalue and assume that*

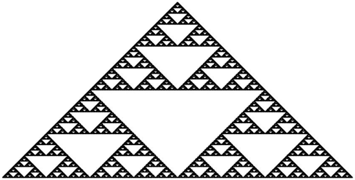
$$\begin{cases} \psi_1([e_3]) = \psi_3([e_1]) = u + iv, \\ \psi_2([e_1]) = \psi_1([e_2]) = is, \\ \psi_3([e_2]) = \psi_2([e_3]) = -u + iv. \end{cases}$$

A necessary condition for f_1, f_2, f_3 *to generate a Sierpinski gasket* symmetric with respect to the imaginary axes, *namely s.t.* $f_1(z) = \overline{f_2(-\bar{z})}$ *and* $f_3(z) = \overline{f_3(-\bar{z})}$, *is that* $\psi_1([e_3]) \in \Gamma$, *where* Γ *is the circle* $x^2 + y^2 - x(1 - s^2) - s^2 = 0$. *For* $s = 0$ *the condition is sufficient for* $u \in [1/5, \alpha]$, *where* $\alpha \simeq 0.651$.

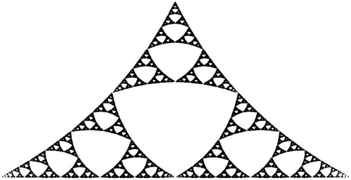
Examples of Symm. Complex proj. Sierpinski gaskets



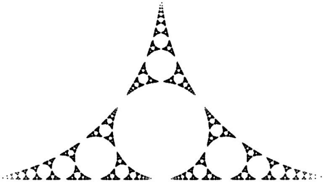
$u = 16/25$, $s_{\boldsymbol{A}} \simeq 2.88$



$u = 1/2$, $s_{\boldsymbol{A}} = 2\log_2 3$



$u = 9/25$, $s_{\boldsymbol{A}} \simeq 2.88$



$u = 1/5$, $s_{\boldsymbol{A}} \simeq 2.61$

All corr. gaskets are fast and, except for the Apollonian gasket, the corr. IFS are hyperbolic, so that $\dim_H \boldsymbol{A} = s_A/2$.

Real projective Sierpinski gaskets

Definition

Let $\boldsymbol{F} = \langle f_1, \ldots, f_n \rangle$ be a free semigroup of volume-preserving linear automorphisms of $\mathbb{R}^n$ with real spectrum and $\psi_1, \ldots, \psi_n \in PSL_n(\mathbb{R})$ the induced projective automorphisms of $\mathbb{R}P^{n-1}$.

Let $\mathcal{E} = \{e_1, \ldots, e_n\}$ be a n-frame of $\mathbb{R}^n$ and $\mathcal{E}^* = \{\varepsilon^1, \ldots, \varepsilon^n\}$ its dual frame. We say that $\boldsymbol{F}$ is a *real projective Sierpinski gasket* over $\mathcal{E}$ if the following conditions are satisfied:

1. $f_i = A^k_{ij} e_k \otimes \varepsilon^j$ with $A^k_{ij} \geq 0$;
2. $f_i(e_i) = \lambda_i e_i$, with $\lambda_i = \max_{1 \leq j \leq n} \{A^j_{ij}\}$;
3. $f_i(e_j) = \alpha e_i + \beta e_j$ with $\alpha, \beta > 0$;
4. $\psi_i([e_j]) = \psi_j([e_i])$, $i, j = 1, \ldots, n$.

Simple real projective Sierpinski gaskets

Definition

We say that a real projective Sierpinski gasket (RPSG) $\boldsymbol{F} = \langle f_1, \dots, f_m \rangle$ is *simple* when each f_i either has only one eigenvalue (first kind) or has exactly two eigenvalues and the eigenspace corresponding to the larger one is 1-dimensional (second kind).

Lemma

The dual semigroup $\boldsymbol{F}^*$ of a simple RPSG is a simple RPSG.

Proposition

Every simple RPSG is a fast gasket.

Examples of simple real projective Sierpinski gaskets

Let $\mathcal{E} = \{e_1, \dots, e_n\}$ a frame of $\mathbb{R}^n$.
The semigroups $\boldsymbol{C}_n^\alpha$ generated by $\alpha^{-1/n} f_i$, with

$$f_i(e_i) = \alpha e_i,\ f_i(e_j) = e_i + e_j,$$

are all simple and so in particular they are all fast and therefore have finite exponent.
They are *hyperbolic* IFS for $\alpha \in (1, 4)$ and parabolic for $\alpha = 1, 4$.

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For $\alpha = 1$, $n = 3$, we get our first motivational example and for $n > 3$ its natural multidimensional generalizations.

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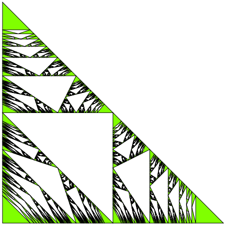
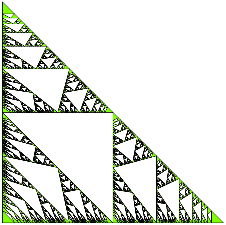
For $\alpha = 1$, $n = 3$, we get our first motivational example and for $n > 3$ its natural multidimensional generalizations.

For $\alpha = 2$, $n = 3$, we get the Sierpinski gasket and for $n > 3$ its natural multidimensional generalizations.

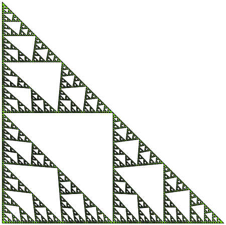
Examples of simple RPGS with $n = 3$



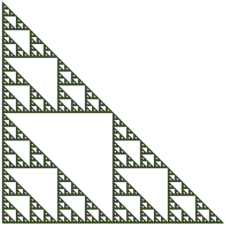
$\alpha = 1, s \simeq 2.444$



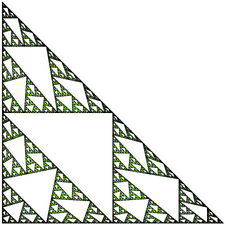
$\alpha = 1.3, s \simeq 2.395$



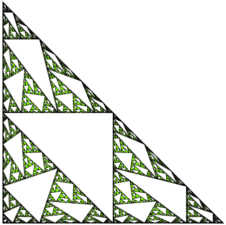
$\alpha = 1.7, s \simeq 2.377$



$\alpha = 2, s = 3 \log_2 3/2$

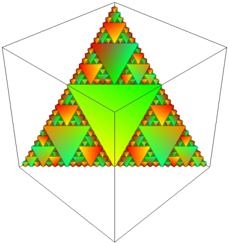


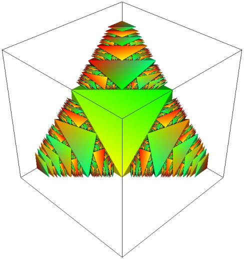
$\alpha = 3, s \simeq 2.378$



$\alpha = 4, s \simeq 2.389$

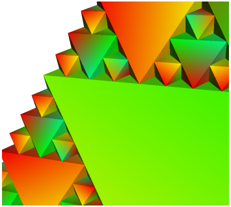
Examples of simple RPGS with $n = 4$

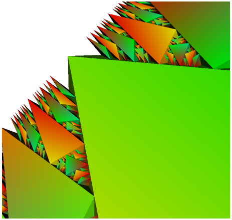




$\alpha = 2$

$\alpha = 1$





A Conjecture on simple RPSG

The problem of determining the Hausdorff dimension of even just self-affine sets is known to be a hard one.
Fundamental contributions to this field are due to K. Falconer[2].

As far as we know nothing is known in case of real self-projective sets for $n \geq 2$.

Based on a few analytical results in particular cases and on numerical experiments we formulate the following conjecture about the set's *Box* dimension:

Conjecture

Let $\mathcal{A} : \mathcal{I}^n \to M_n(\mathbb{R})$, $n \geq 3$, be a simple RPSG.
Then $n \dim_B R_A \geq (n-1) s_A$.

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[2] K.J. Falconer, The Hausdorff dimension of self-affine fractals, Math. Proc. Camb. Phil. Soc. 103 (1988)

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