Topology and Dynamics of Quasiperiodic Functions

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Outline

1. Quick history of QuasiPeriodic Functions (QPFs)

2. Definitions of Almost- and Quasi-Periodic functions

3. QPFs in Analysis, Geometry & Dyn. Systems

4. Magnetoresistance & QPFs in $\mathbb{R}^2$ with 3 quasiperiods

A few historical remarks

QuasiPeriodic functions in one variable were introduced in literature by the Latvian mathematician P. Bohl in 1893.

The term is due to the French astronomer E. Esclangon in 1904.

The first big works on the topic are due to H. Bohr (stronger soccer player than his brother Niels!) that founded and developed the theory of Almost Periodic Functions in several works since 1925.

The first to consider AP functions in $\mathbb{R}^n$, $n > 1$, was S. Bochner in 1927.

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Almost Periodic Functions

Definition

$AP(\mathbb{R}^k)$ is the closure, in $L^\infty(\mathbb{R}^k)$, of the subalgebra generated by all functions $e_\lambda(x) = e^{i\langle\lambda,x\rangle}$, $\lambda \in (\mathbb{R}^k)^*$.

The Bohr-Fourier coefficients of $f \in AP(\mathbb{R}^k)$ are the

$$f_\lambda = \lim_{T\to\infty} \frac{1}{(2T)^k} \int_{[-T,T]^k} e^{-i\langle\lambda,x\rangle} f(x)dx\,,\ \lambda \in (\mathbb{R}^k)^*$$

The Bohr-Fourier spectrum of f is the free abelian group

$$\sigma_f = \langle \lambda \in (\mathbb{R}^k)^* \mid f_\lambda \neq 0 \rangle$$

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A characterization of APFs

A function $f \in L^\infty(\mathbb{R}^n)$ is almost continuous if and only if for every $\epsilon > 0$ it has a relatively dense set $P_\epsilon \subset \mathbb{R}^n$ of "ϵ-periods", namely of vectors $\tau \in \mathbb{R}^n$ such that

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |f(x+\tau) - f(x)| \leq \epsilon$$

[Recall that *relatively dense* means that the set has no gaps of arbitrary diameter]

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Quasiperiodic Functions

Analytical Definition

A $f \in AP(\mathbb{R}^k)$ is called quasiperiodic if σ_f is finitely generated.

Geometrical Definition

Let $\pi_n : \mathbb{R}^n \to \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the canonical projection from the n-Euclidean space to the n-torus.

Then $f \in L^\infty(\mathbb{R}^k)$ is quasiperiodic if

$$f = F \circ \pi_n \circ \psi$$

for some $F \in L^\infty(\mathbb{T}^n)$ and an affine embedding $\psi : \mathbb{R}^k \to \mathbb{R}^n$,

namely if f is the restriction of a n-periodic function F to a k-plane $\psi(\mathbb{R}^k) \subset \mathbb{R}^n$.

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Quasiperiods

When f is quasiperiodic, the number of generators of σ_f is called the number of *quasiperiods* of f.

E.g. $f(x) = \cos(2\pi x) + \cos(2\sqrt{2}\pi x) + \cos(2(1+\sqrt{2})\pi x)$
is a quasiperiodic function in 1 variable with 2 quasiperiods
since $\sigma_f = \langle 1, \sqrt{2}, 1+\sqrt{2}\rangle$

while
$g(x,y) = \cos(2\pi x) + \cos(2\pi y) + \cos(2\pi(\sqrt{2}x + \sqrt{3}y + \sqrt{5}))$
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Quasiperiodic Functions in one variable: Hamiltonian systems [Liouville]

Since XIX century, quasiperiodic functions in one variable appeared naturally in the theory of completely integrable Hamiltonian systems.

E.g. it is well known that, by the Liouville's theorem, when a Hamiltonian system with n degrees of freedom has n independent pairwise first integrals, if their common level set is compact, then the solutions of the equations of motion are quasiperiodic functions of time with *at most* n quasiperiods.

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Quasiperiodic Functions in *more than* one variable: completely integrable PDEs [Novikov]

S.P. Novikov was the first to find concrete cases where arise quasiperiodic functions with more than one variables.

In Seventies, he discovered that many completely integrable PDEs admit quasiperiodic functions in k variables, with $k > 1$. For example, the KdV eq. $u_t = 6uu_x + u_{xxx}$ admits solutions $u : \mathbb{R}^2 \to \mathbb{R}$ of the form

$$u(x,t)_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}} = F(x\boldsymbol{a} + t\boldsymbol{b} + \boldsymbol{c}) ,$$

where $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^n$, $F : \mathbb{T}^n \to \mathbb{R}$ is a n-periodic function that can be written in terms of theta-function of a hyperelliptic Riemann surface of genus n and $\boldsymbol{a}, \boldsymbol{b}$ are the vectors of periods of some Abelian differential of the second kind on that surface.

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Quasiperiodic Functions in *more than* one variable: Quasicrystals [Novikov]

Quasicrystals tilings of $\mathbb{R}^k$ are a collection of a countable collection of closed polytopes whose union is the whole $\mathbb{R}^k$, whose pairwise intersection is either empty or an entire subpolytope and s.t., modulo translations, there is only a *finite* number of them.

They were discovered in nature in Eighties by D. Schechtman, Nobel prize in 2011 for this discovery, and are stricly related to QP functions in the following way:

Let $P_1, \dots, P_N$ be these polytopes and consider any piecewise continuous function constant on the interior of each P_i. Then f is quasiperiodic.

[LeTu, Piunikin, Sadov, *The geometry of quasicrystals*, Russian Mathematical Surveys 48:1, 1993]

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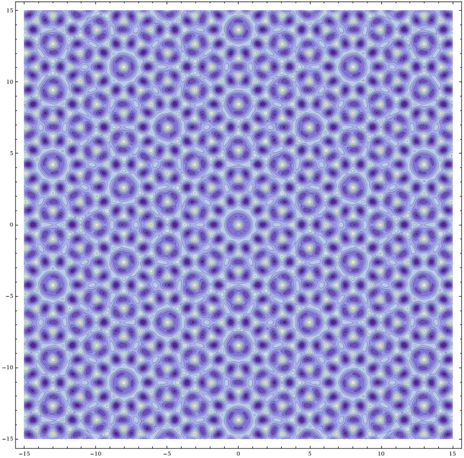
Example of (almost) quasycrystal

Section of $\sum_{i=1}^{5}\cos(2\pi x_i)=0$ through the 2-plane spanned by the real and imaginary part of $(e^{2\pi i/5},\dots,e^{2\pi 5i/5})$.



Quasiperiodic Functions in *more than* one variable: Poisson Dynamics [Novikov]

Consider the Poisson manifold $(\mathbb{T}^3, \{\}_B)$ where, in coords (p_1, p_2, p_3) for $\mathbb{T}^3$,

$$\{p_i, p_j\}_B = \epsilon_{ijk} B^k$$

for some constant 1-form $B = B^k dp_k$.

Since we are in odd dimension, necessarily this Poisson braket has (at least) one Casimir (i.e. a function that commutes with *any* other function). This Casimir is the (multivalued!) function

$$b(p) = B^k p_k + b_0 \,.$$

Note that $db = B$ but B is not exact (in $\mathbb{T}^3$) because $b(p)$ is not single-valued. It is easy to check that $\{,\}_B$ is non-degenerate on every plane $b(p) = const$, namely there are no other Casimirs.

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$$\dot{p} = \{p, H(p)\}_B ,$$

describing the *image* of such solutions, namely their *orbits*, it is trivial: they are the intersections of level surfaces of H with planes perpendicular to B.

In other words, they are the level sets of quasiperiodic functions in two variables and (at most) three quasiperiods, namely the restrictions of H to planes perpendicular to B.

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Magnetoresistance in normal metals

Novikov noticed that the previous problem actually comes from Solid State Physics: in the semiclassical model, electrons' quasi-momenta are *periodic* and the equations of motion, in case of a constant magnetic field B, are given by

$$\dot{p} = \frac{e}{c}\nabla\varepsilon \times B\,,$$

where ε is the Fermi Energy function.

Note that, in coordinates, this is exactly the same system of equations of the Poisson bracket in the previous slide with $H = \varepsilon$.

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It was pointed out by Lifshitz, Azbel and Kaganov in Fifties that, in the semiclassical approximation, the qualitative behavior of the magnetoresistance depends on the *topology* of the level sets of the restriction of ε to planes perpendicular to B, namely:

1. it saturates if only closed level sets arise;
2. it grows monotonically with the intensity of B if open orbits arise.

In other words, the *topology* of these level sets is *observable*.

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This prediction, and threfore a strong confirmation of the effectiveness of the semiclassical approximation, was soon found experimentally by Pippard in Copper and by Gaidukov and Alekseevski in Silver, Gold and several other metals.

Next picture, by Gaidukov (1960), shows $\mathbb{R}P^2$ as a disc (with opposite bd pts identified). The shaded regions are filled by directions of the magnetic field giving rise to asymptotics of the magnetoresistance that correspond, in the semiclassical approximation, to the presence of open orbits for the quasimomenta. The unshaded area correspond to all closed orbits.

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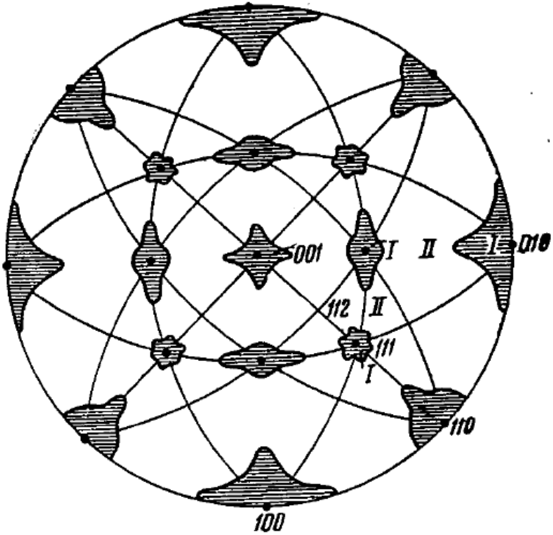

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The interest in this topic faded after a decade since no way was found to answer the following questions:

1. Why are the B giving rise to open orbits sorted in "islands"?
2. Is this the generic picture to be expected?
3. Is there any difference between B belonging to different islands?
4. How to predict the distribution of the islands from the Fermi Energy function ε?

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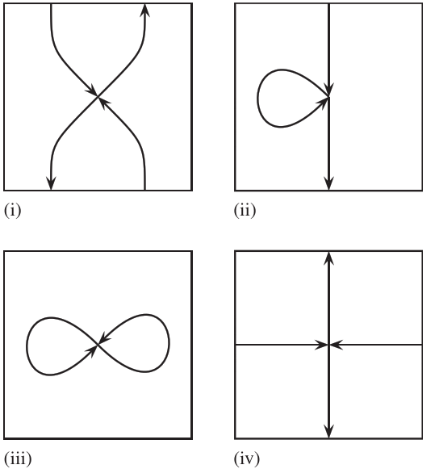
Magnetoresistance in normal metals

The interest in this topic faded after a decade since no way was found to answer the following questions:

1. Why are the B giving rise to open orbits sorted in "islands"?
2. Is this the generic picture to be expected?
3. Is there any difference between B belonging to different islands?
4. How to predict the distribution of the islands from the Fermi Energy function ε?

The close-to-rational case [Zorich, 1983]



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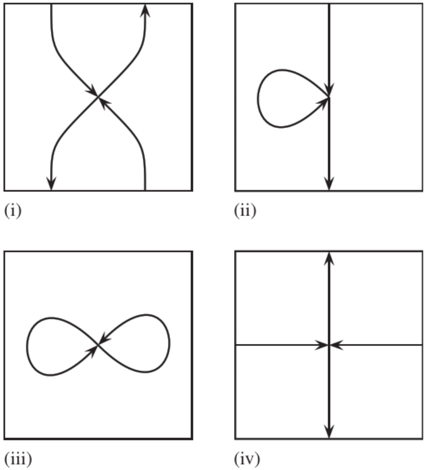

Only (i) and (ii) involve open orbits and are compatible with the boundary conditions. Since (i) is not stable by small perturbations, (ii) is the only relevant saddle type for this problem.

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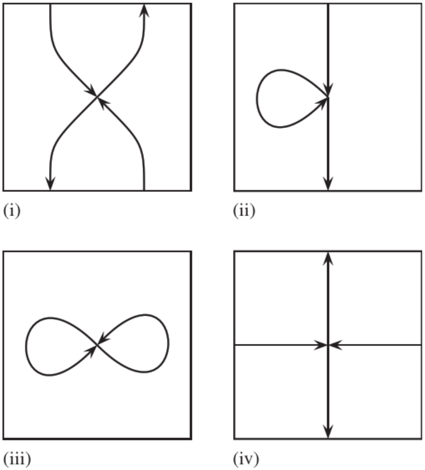
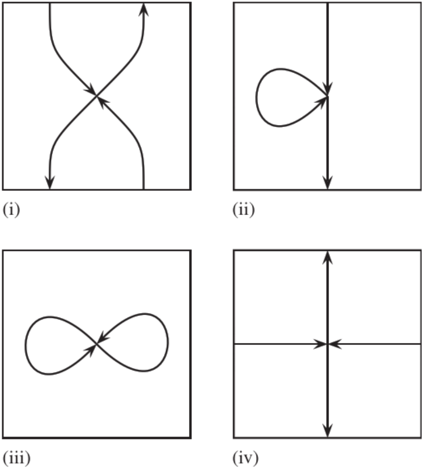


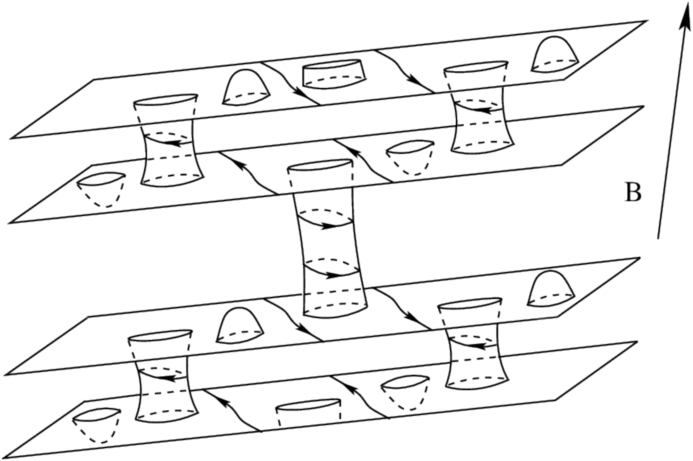
Hence open orbits are contained in components of the Fermi Surface with genus 1. Since loops homotopic to zero are stable by small perturbations, this is true also for all directions close to rational!

The close-to-rational case [Zorich, 1983]



These 2-tori filled by open orbits have all the same homology class (modulo sign) in $H_2(\mathbb{T}^3, \mathbb{Z}) \simeq \mathbb{Z}^3$. This is a *hidden quantum first integral* of the system and it completely describes the asymptotics of the open orbits.

Same holds in the generic case [Dynnikov, 1992]



Structure of level sets of QPFs on $\mathbb{R}^2$ with 3 quasiperiods

Let $f(x,y) = F(x,y,\alpha x + \beta y + \gamma)$, where $F(x,y,z)$ is periodic in x,y,z and $B = (\alpha,\beta,-1)$. Assume that B is *generic*, namely that $1,\alpha,\beta$ are *rationally independent*. What can we say about the level sets of f?

First, open level sets of f do arise *only* for a closed interval of values $[l_F(B), u_F(B)]$ *that depends continuously on* B.

Moreover, in the most interesting case, the topological first integral is defined for an open dense set of directions $B \in \mathbb{R}P^2$ sorted in disjoint "islands" $\mathcal{D}_\ell$ that are labeled by different elements $\ell \in H_2(\mathbb{T}^3,\mathbb{Z})$ and meet transversally to each other like in the Sierpinsky gasket fractal.

If $B \in \mathcal{D}_\ell$, then all non-singular open orbits are strongly asymptotic to a straight line with direction "$B \times \ell$".

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Quasiperiodic Functions

R. De Leo

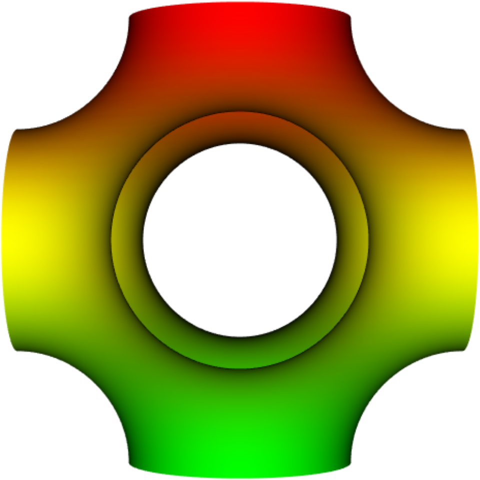
Prehistory

Definitions

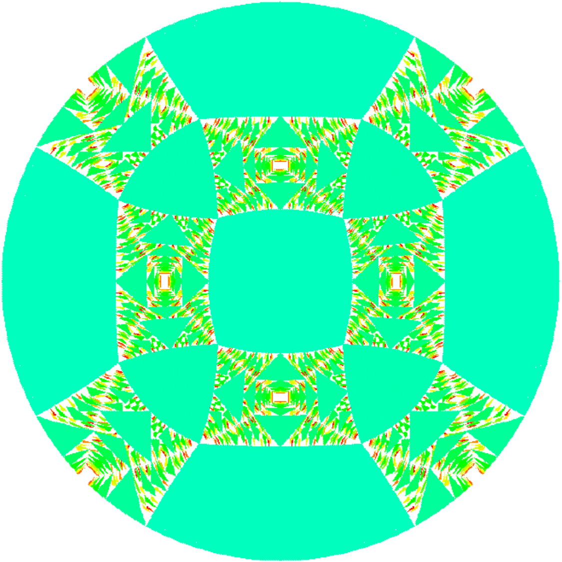
QPFs in Analysis, Geometry & Dyn. Systems

QPFs in $\mathbb{R}^2$ with 3 quasiperiods

Example: $F(x, y, z) = \cos(2\pi x) + \cos(2\pi y) + \cos(2\pi z)$

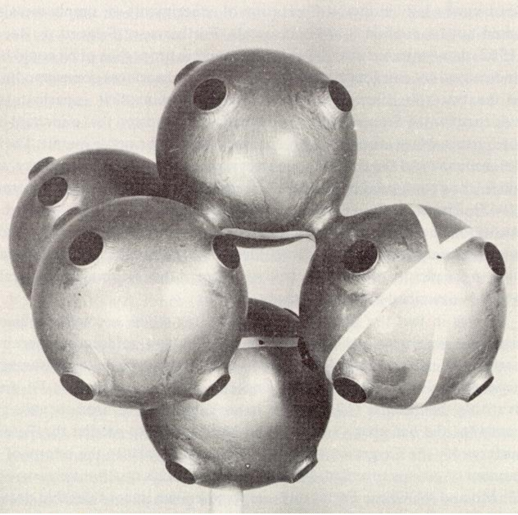


Example: $F(x, y, z) = \cos(2\pi x) + \cos(2\pi y) + \cos(2\pi z)$



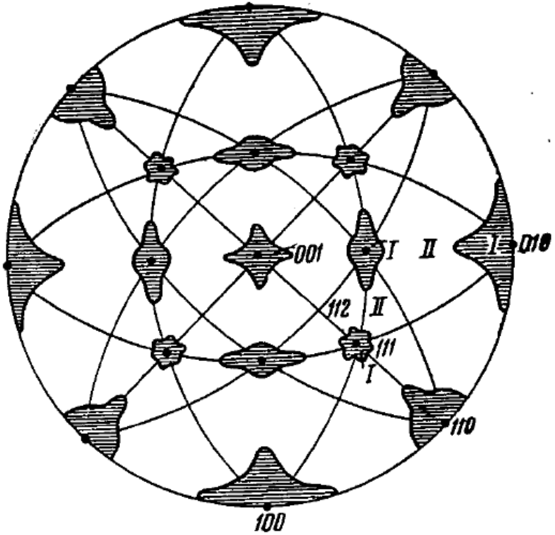
Comparison with Au experimental data

Fermi Surface of Au



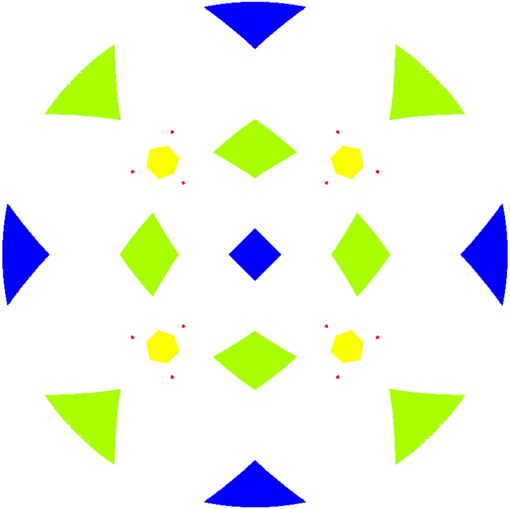
Comparison with Au experimental data

Experimental data for Au's magnetoresistance



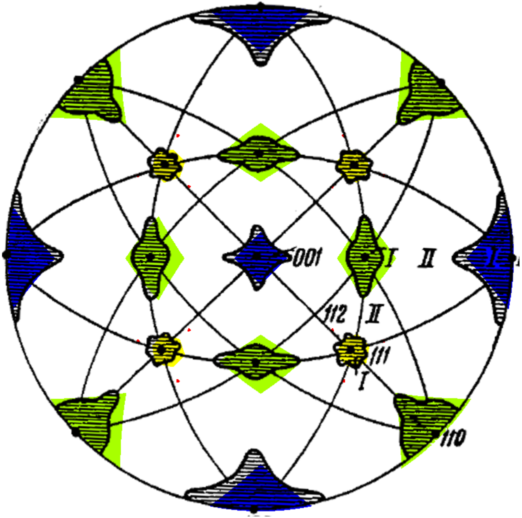
Comparison with Au experimental data

Numerical data for Au's magnetoresistance



Comparison with Au experimental data

Matching of the experimental and numerical data



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3. R. De Leo, *First-principles generation of Stereographic Maps for high-field magnetoresistance in normal metals: an application to Au and Ag*, Physica B 362, 2005, cond-mat/0409383
4. A. Ya. Maltsev, S.P. Novikov, *The theory of closed 1-forms, levels of quasiperiodic functions and transport phenomena in electron systems*, arXiv:1805.05210
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