Exponential growth of norms in semigroups of linear automorphisms and Hausdorff dimension of self-projective iterated function systems

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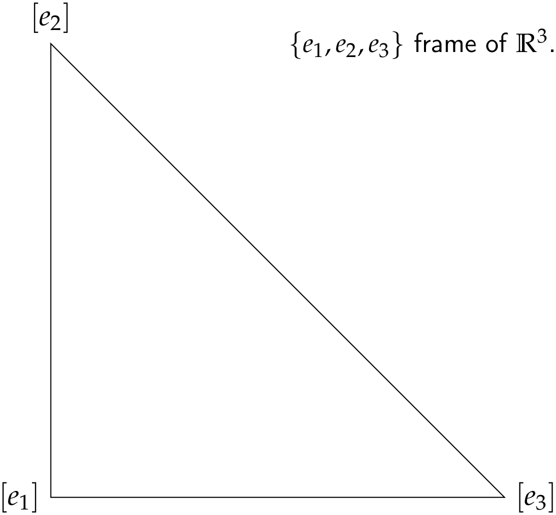
The Levitt-Yoccoz gasket



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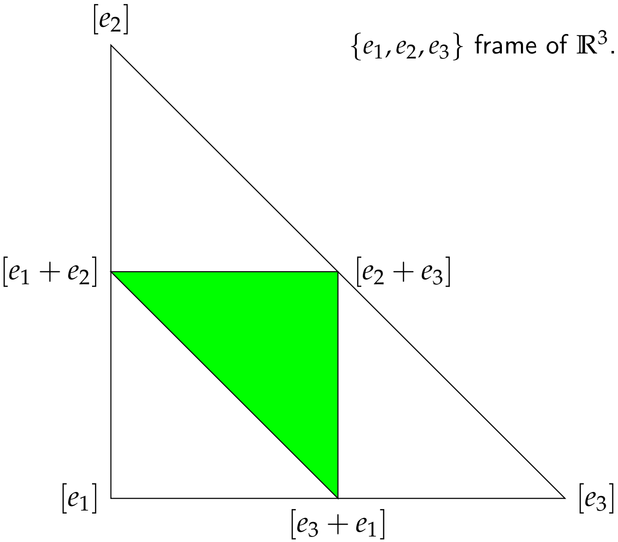

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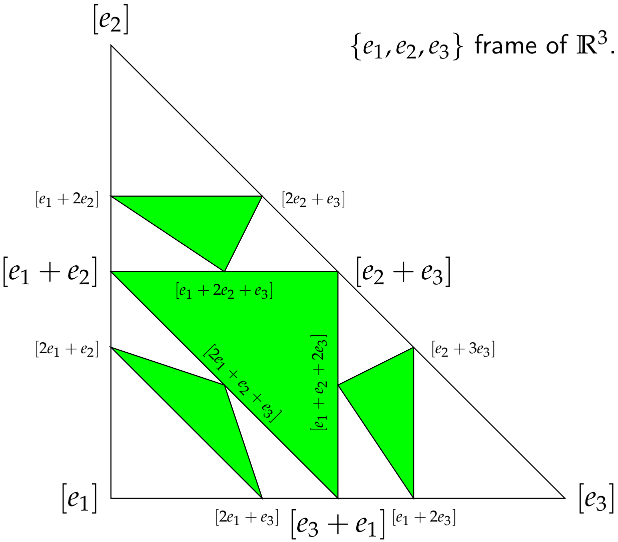

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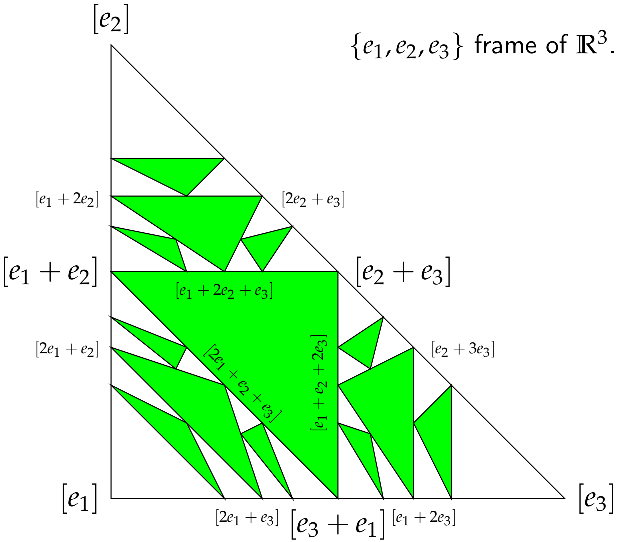

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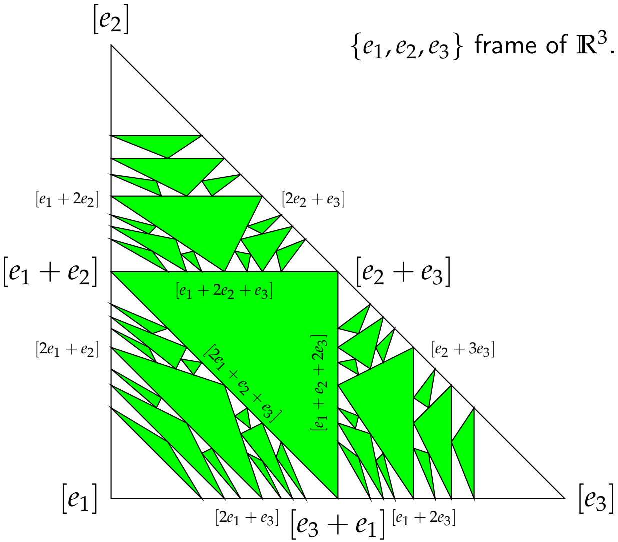

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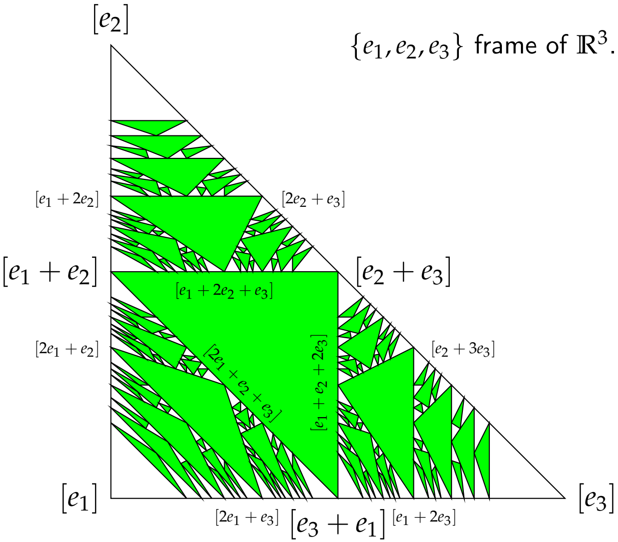

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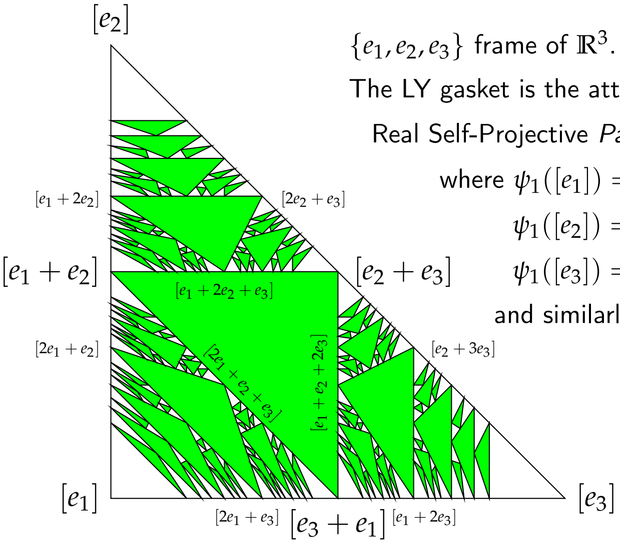

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The LY gasket is the attractor of the

Real Self-Projective *Parabolic* IFS $\{\psi_1, \psi_2, \psi_3\}$,

where $\psi_1([e_1]) = [e_1]$,

$\psi_1([e_2]) = [e_2 + e_1]$,

$\psi_1([e_3]) = [e_3 + e_1]$,

and similarly for ψ_2, ψ_3.

The Levitt-Yoccoz gasket – references

- P. Arnoux, G. Rauzy, *Représentations géométriques de suites de complexité* $2n+1$, Bull. Soc. Math. France, **1991**

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The Semigroup C

The self-projective automorphisms ψ_1, ψ_2, ψ_3
are induced,w/resp to the frame $\{e_1, e_2, e_3\}$, by the linear maps

$$\begin{matrix} & C_1 & C_2 & C_3 & \\ C = \Bigg\langle & \begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 0 & 1\end{pmatrix}, & \begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}, & \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix} & \Bigg\rangle \subset SL_3(\mathbb{N})\end{matrix}$$

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Note that $\sigma(C_i) = \{1\}$,
so $\|C_i^k\|$ grows *polynomially*

e.g. $C_1^k = \begin{pmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ k & 0 & 1 \end{pmatrix}$

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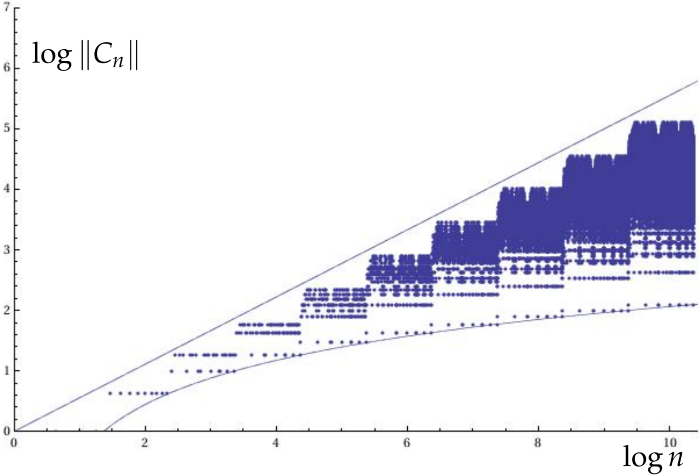
While, for $i \neq j$, $\max\limits_{\lambda \in \sigma(C_i C_j)} |\lambda| > 1$,
so $\|C_i C_j\|^k$ grows exponentially

e.g. $C_1 C_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix}$

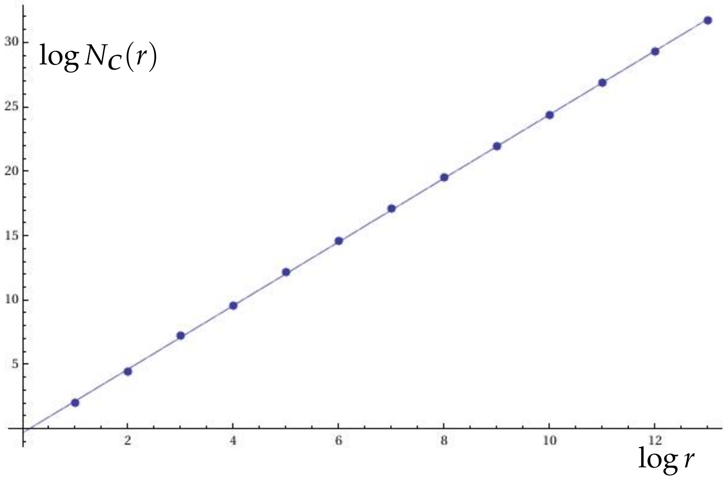
Norm Asymptotics in C – lexicographic order

Log-log plot of norms of elements of C in lexicographic order:



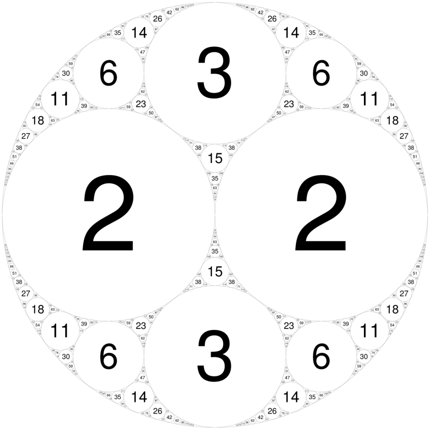
Norm Asymptotics in C – non-decreasing order

Log-log plot of norms of elements of C in non-decreasing order:



Radii Asymptotics in the Apollonian Gasket A

This behaviour is not uncommon, for example it is shared by the celebrated Apollonian gasket A:



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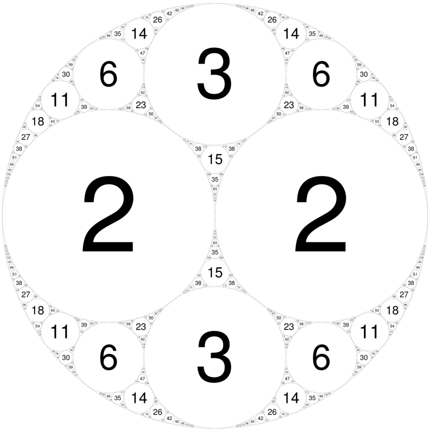

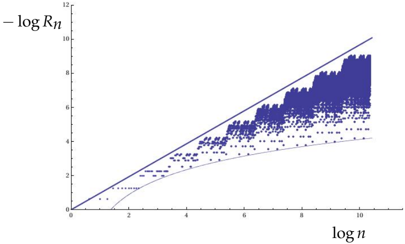

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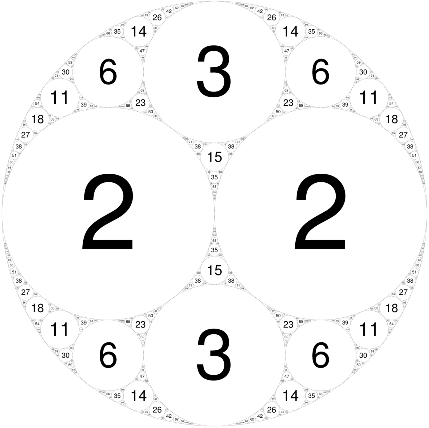

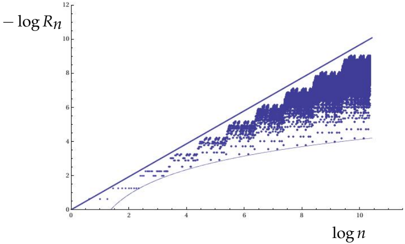

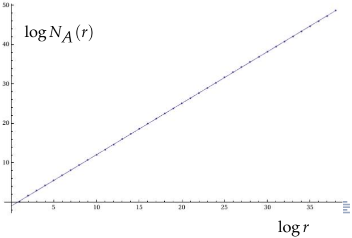

The Semigroup $\boldsymbol{H}$

The radii^{-1} of circles in $\boldsymbol{A}$ grow like the norms of the *Hirst* matrices

$$\boldsymbol{H} = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix} \right\rangle \subset SL_4(\mathbb{N})$$

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In a series of papers in 70s D. Boyd[1] proved (geometrically) that

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In 2011 Kontorovich & Oh[2] proved (again geometrically) the stronger result $N_{\boldsymbol{H}}(r) \asymp r^d$.

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Natural Questions

- Do really the norms in C grow exponentially as suggested by the numerics? Namely, does $\lim_{r\to\infty} \frac{\log N_C(r)}{\log r}$ really converges to some $s_C \in \mathbb{R}^+$?

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- Is it true in general that $N_S(r) \asymp r^{s_S}$?

Main Results

Definition

A finitely generated semigroup $S = \langle A_1, \ldots, A_m \rangle \subset SL_n(K)$, $K = \mathbb{R}, \mathbb{C}$, is *fast* if there is a $c > 0$ s.t.

$$\|A_{IJK}\| \geq c\|A_I\|\,\|A_{JK}\|$$

for all multi-indices I, J, K, with $J = j_1 \ldots j_k$ s.t. $j_1 \neq j_2 = \cdots = j_k$.

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Theorem (RdL, 2012)

Let $S = \langle A_i \rangle$ be a free fast subsemigroup of $SL_n(K)$, $K = \mathbb{R}, \mathbb{C}$.

Then $\lim_{r\to\infty} \frac{\log N_S(r)}{\log r}$ converges to a finite $s_S > 0$. Moreover,

$$s_S = \sup_{s\geq 0}\{s | \sum_I \|A_I\|^{-s} = \infty\} = \inf_{s\geq 0}\{s | \sum_I \|A_I\|^{-s} < \infty\}.$$

Example

$$S = \left\langle A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle \subset SL_2(\mathbb{N})$$

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$$\|A_{IJK}\| \geq \|A_I\| \max\{a + kc, b + kd\} \geq \frac{k}{k+1} \|A_I\| \|A_{JK}\| \geq \frac{1}{2} \|A_I\| \|A_{JK}\|$$

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It can be proved similarly that C and H are parabolic fast gaskets.

Norms growth and Hausdorff dimension of IFS attractors

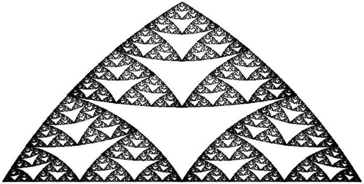
Theorem (RdL, 2012)

Let $\{A_1, \ldots, A_m\} \subset SL_2(K), K = \mathbb{R}, \mathbb{C}$, *and denote by* $\psi_i \in PSL_2(K)$ *the projective automorphism associated to* f_i.
Assume that the A_i *are all hyperbolic and that there exists some proper open set* $V \subset \mathbb{R}P^1$ *(resp.* $V \subset \mathbb{C}P^1$*) invariant under the* ψ_i *such that, for some affine chart* $\varphi : \mathbb{R}P^1 \to \mathbb{R}$ *(resp. some complex affine chart* $\varphi : \mathbb{C}P^1 \to \mathbb{C}$*), the* ψ_i*:*

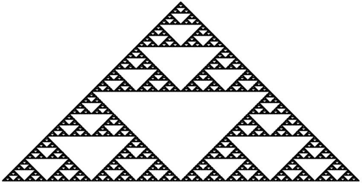
1. *are contractions on* $\varphi(\overline{V})$ *with respect to the Euclidean distance;*
2. *satisfy* $0 < a \leq |\psi_i'(v)| \leq c < 1$ *for all* $1 \leq i \leq m$, $v \in V$ *and some constants* a, c;
3. *satisfy the open set condition* $i \neq j \implies \psi_i(V) \cap \psi_j(V) = \emptyset$.

Let $R_{\mathbf{A}} = \cap_{k=1}^{\infty} \left(\cup_{|I|=k} \psi_I(V) \right)$ *be the corresponding attractor.*
Then $2 \dim_H R_{\mathbf{A}} = s_{\mathbf{A}}$.

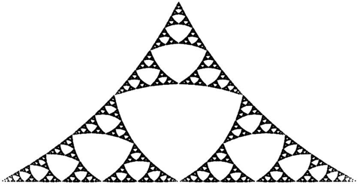
Example



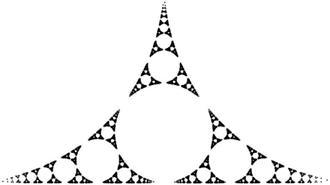
a



b



c



d

State of the art

- The property of a free semigroup $S \subset SL_n(K), K = \mathbb{R}, \mathbb{C}$ of being *fast* is sufficient to grant that $\lim_{r \to \infty} \frac{\log N_S(r)}{\log r} = s_S < \infty$.

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- We provided a constructive iterative algorithm to evaluate analytical upper and lower bounds for s_S with any degree of accuracy (logarithmic speed).
- The exponent s_S completely determines the Hausdorff dimension of the attractor of the self-projective IFS induced by S for $n = 2$.

Open Questions

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Conjecture (RdL, 2012)

$(n+1)\dim_H R_S \geq n s_S$

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Thanks!