Asymptotics of plane sections of the regular skew polyhedron $\{4,6\,|\,4\}$.

I.A. Dynnikov

Dept. of Mechanics and Mathematics, Moscow State University

R. De Leo

Dept. of Physics, U. of Cagliari & Nat.Inst. for Nucl. Phys., Cagliari

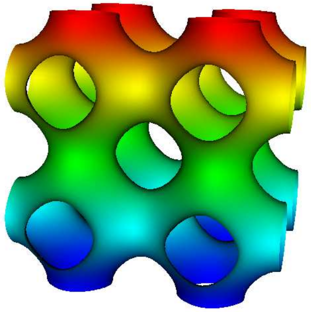
19 April 2007

With Acroread, CTRL-L switch between full screen and window mode

1 – Statement of the problem I - Elementary geometry

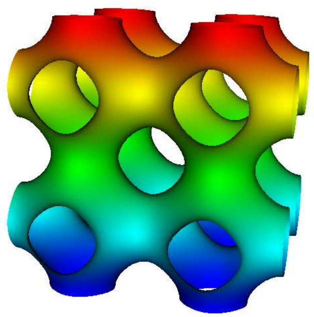
1 – Statement of the problem I - Elementary geometry

Consider a 3-ply periodic surface in $\mathbb{R}^3$



1 – Statement of the problem I - Elementary geometry

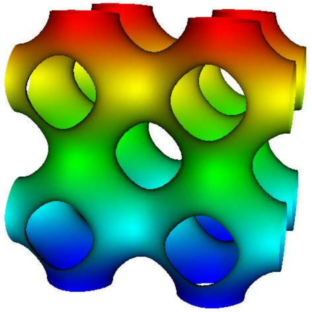
Consider a 3-ply periodic surface in $\mathbb{R}^3$



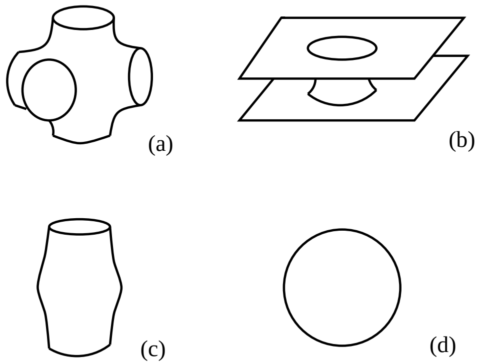
and cut it with a bundle of parallel planes.

1 – Statement of the problem I - Elementary geometry

Consider a 3-ply periodic surface in $\mathbb{R}^3$



and cut it with a bundle of parallel planes.



Q: what can be said about the asymptotics of the open intersections?

2 – Statement of the problem II - Foliations of surfaces

2 – Statement of the problem II - Foliations of surfaces

$i : \mathcal{M}^2_g \to \mathbb{T}^3$

rk$(i) = 3$

rk(i) is the rank of

$i_* : H_1(\mathcal{M}^2_g, \mathbb{Z}) \to H_1(\mathbb{T}^3, \mathbb{Z})$

2 – Statement of the problem II - Foliations of surfaces

$$i:\mathcal{M}^2_g \to \mathbb{T}^3$$
$$\text{rk}(i)=3$$

rk(i) is the rank of
$$i_*:H_1(\mathcal{M}^2_g,\mathbb{Z})\to H_1(\mathbb{T}^3,\mathbb{Z})$$

$$\Omega=\Omega_x dx+\Omega_y dy+\Omega_z dz$$
constant 1-form

$$\omega=i^*\Omega\in\Omega^1(\mathcal{M}^2_g)$$
is a closed 1-form

2 – Statement of the problem II - Foliations of surfaces

$i : \mathcal{M}^2_g \to \mathbb{T}^3$

rk$(i) = 3$

rk(i) is the rank of

$i_* : H_1(\mathcal{M}^2_g, \mathbb{Z}) \to H_1(\mathbb{T}^3, \mathbb{Z})$

$\Omega = \Omega_x dx + \Omega_y dy + \Omega_z dz$

constant 1-form

$\omega = i^* \Omega \in \Omega^1(\mathcal{M}^2_g)$

is a closed 1-form

Q: what can be said about asymptotics

of non-compact leaves of ω on $\mathcal{M}^2_g$?

For a generic closed 1-form ω non compact leaves are dense on the surface

for this restricted class the situation is actually totally reversed.

3 – Statement of the problem III - Multivalued Poisson dynamical systems

3 – Statement of the problem III - Multivalued Poisson dynamical systems

Consider the Poisson structure

$$\{p_i, p_j\}_{\Omega} = \epsilon_{ijk} p_i p_j \Omega_k$$

on $\mathbb{T}^3$

Take a Hamiltonian

$$H : \mathbb{T}^3 \to \mathbb{R}$$

3 – Statement of the problem III - Multivalued Poisson dynamical systems

Consider the Poisson structure

$$\{p_i, p_j\}_{\Omega} = \epsilon_{ijk} p_i p_j \Omega_k$$

on $\mathbb{T}^3$

Take a Hamiltonian

$$H : \mathbb{T}^3 \to \mathbb{R}$$

$\{,\}_{\Omega}$ has a Casimir $f_{\Omega} = \Omega_x x + \Omega_y y + \Omega_z z$, i.e. $\{f_{\Omega}, g\}_{\Omega} = 0 \quad \forall g \in C^{\infty}(\mathbb{T}^3)$

3 – Statement of the problem III - Multivalued Poisson dynamical systems

Consider the Poisson structure

$$\{p_i, p_j\}_{\boldsymbol{\Omega}} = \epsilon_{ijk} p_i p_j \Omega_k$$

on $\mathbb{T}^3$

Take a Hamiltonian

$$H : \mathbb{T}^3 \to \mathbb{R}$$

$\{,\}_{\boldsymbol{\Omega}}$ has a Casimir $f_{\boldsymbol{\Omega}} = \Omega_x x + \Omega_y y + \Omega_z z$, i.e. $\{f_{\boldsymbol{\Omega}}, g\}_{\boldsymbol{\Omega}} = 0 \quad \forall g \in C^\infty(\mathbb{T}^3)$

Trajectories therefore are defined by $H = e_0$, $f_{\boldsymbol{\Omega}} = f_0$.

3 – Statement of the problem III - Multivalued Poisson dynamical systems

Consider the Poisson structure

$$\{p_i, p_j\}_\Omega = \epsilon_{ijk} p_i p_j \Omega_k$$

on $\mathbb{T}^3$

Take a Hamiltonian

$$H : \mathbb{T}^3 \to \mathbb{R}$$

$\{,\}_\Omega$ has a Casimir $f_\Omega = \Omega_x x + \Omega_y y + \Omega_z z$, i.e. $\{f_\Omega, g\}_\Omega = 0 \quad \forall g \in C^\infty(\mathbb{T}^3)$

Trajectories therefore are defined by $H = e_0$, $f_\Omega = f_0$.

Note that f_Ω is multivalued on $\mathbb{T}^3$

3 – Statement of the problem III - Multivalued Poisson dynamical systems

Consider the Poisson structure

$$\{p_i,p_j\}_\Omega = \epsilon_{ijk}p_ip_j\Omega_k$$

on $\mathbb{T}^3$

Take a Hamiltonian

$$H:\mathbb{T}^3\to\mathbb{R}$$

$\{,\}_\Omega$ has a Casimir $f_\Omega = \Omega_x x + \Omega_y y + \Omega_z z$, i.e. $\{f_\Omega, g\}_\Omega = 0 \quad \forall g \in C^\infty(\mathbb{T}^3)$

Trajectories therefore are defined by $H = e_0$, $f_\Omega = f_0$.

Note that f_Ω is multivalued on $\mathbb{T}^3$
but its differential $df_\Omega = \Omega$ is well defined.

3 – Statement of the problem III - Multivalued Poisson dynamical systems

Consider the Poisson structure

$$\{p_i, p_j\}_{\Omega} = \epsilon_{ijk} p_i p_j \Omega_k$$

on $\mathbb{T}^3$

Take a Hamiltonian

$$H : \mathbb{T}^3 \to \mathbb{R}$$

$\{,\}_{\Omega}$ has a Casimir $f_{\Omega} = \Omega_x x + \Omega_y y + \Omega_z z$, i.e. $\{f_{\Omega}, g\}_{\Omega} = 0 \;\; \forall g \in C^{\infty}(\mathbb{T}^3)$

Trajectories therefore are defined by $H = e_0$, $f_{\Omega} = f_0$.

Note that f_{Ω} is multivalued on $\mathbb{T}^3$
but its differential $df_{\Omega} = \Omega$ is well defined.

A topological invariant arises from this dynamical system.

4 – The topological invariant

Theorem [Dynnikov] Be H a generic real function on $\mathbb{T}^3$.

Then there exist two continous functions $e_{1,2} : \mathbb{R}P^2 \to \mathbb{R}$ s.t.

4 – The topological invariant

Theorem [Dynnikov] Be H a generic real function on $\mathbb{T}^3$.
Then there exist two continous functions $e_{1,2} : \mathbb{R}P^2 \to \mathbb{R}$ s.t.

- $e_1 \leq e_2$;

4 – The topological invariant

Theorem [Dynnikov] Be H a generic real function on $\mathbb{T}^3$.
Then there exist two continous functions $e_{1,2} : \mathbb{R}P^2 \to \mathbb{R}$ s.t.

- $e_1 \le e_2$;
- open trajectories arise on $\mathcal{M}_e = H^{-1}(e)$ iff $e \in [e_1, e_2]$;

4 – The topological invariant

Theorem [Dynnikov] Be H a generic real function on $\mathbb{T}^3$.
Then there exist two continous functions $e_{1,2} : \mathbb{R}P^2 \to \mathbb{R}$ s.t.

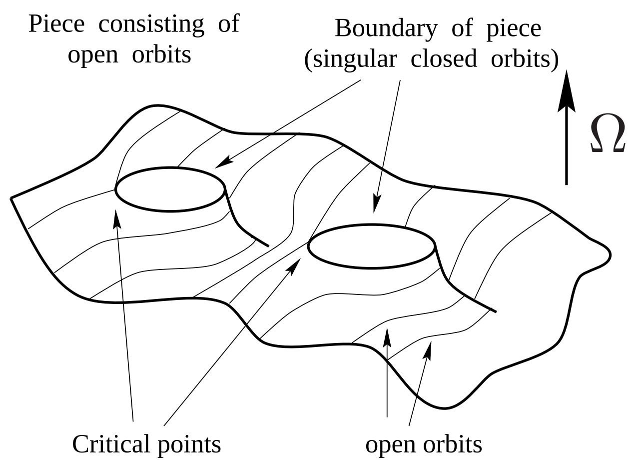
- $e_1 \leq e_2$;
- open trajectories arise on $\mathcal{M}_e = H^{-1}(e)$ iff $e \in [e_1, e_2]$;
- if $e_1(\Omega) \neq e_2(\Omega)$ then every open trajectory fills some g-1 rk-2 component $\mathcal{N}_i$ of $\mathcal{M}_e$.

4 – The topological invariant

Theorem [Dynnikov] Be H a generic real function on $\mathbb{T}^3$.
Then there exist two continous functions $e_{1,2} : \mathbb{R}P^2 \to \mathbb{R}$ s.t.

- $e_1 \le e_2$;
- open trajectories arise on $\mathcal{M}_e = H^{-1}(e)$ iff $e \in [e_1, e_2]$;
- if $e_1(\Omega) \ne e_2(\Omega)$ then every open trajectory fills some g-1 rk-2 component $\mathcal{N}_i$ of $\mathcal{M}_e$.

All $\{\hat{\mathcal{N}}_i\}$ share the same hom class $l = [\hat{\mathcal{N}}_i] \in H_2(\mathbb{T}^3, \mathbb{Z})$



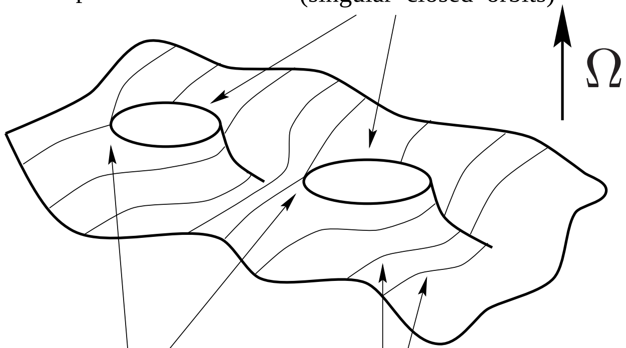
4 – The topological invariant

Theorem [Dynnikov] Be H a generic real function on $\mathbb{T}^3$.
Then there exist two continous functions $e_{1,2} : \mathbb{R}P^2 \to \mathbb{R}$ s.t.

- $e_1 \le e_2$;
- open trajectories arise on $\mathcal{M}_e = H^{-1}(e)$ iff $e \in [e_1, e_2]$;
- if $e_1(\Omega) \ne e_2(\Omega)$ then every open trajectory fills some g-1 rk-2 component $\mathcal{N}_i$ of $\mathcal{M}_e$.

All $\{\hat{\mathcal{N}}_i\}$ share the same hom class $l = [\hat{\mathcal{N}}_i] \in H_2(\mathbb{T}^3, \mathbb{Z})$

l is the same for all $e \in [e_1(\Omega), e_2(\Omega)]$



4 – The topological invariant

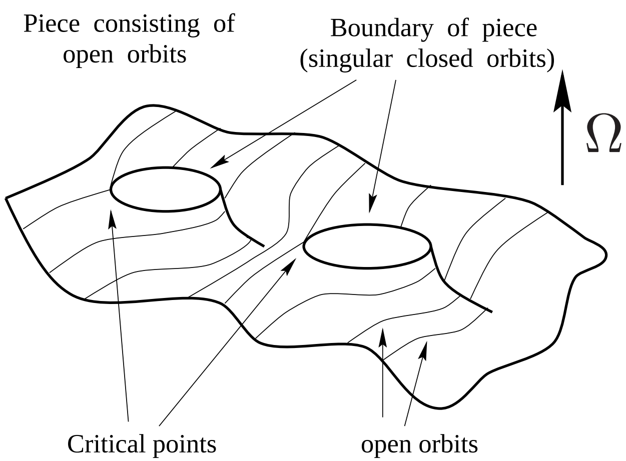
Theorem [Dynnikov] Be H a generic real function on $\mathbb{T}^3$.
Then there exist two continous functions $e_{1,2} : \mathbb{R}P^2 \to \mathbb{R}$ s.t.

- $e_1 \leq e_2$;
- open trajectories arise on $\mathcal{M}_e = H^{-1}(e)$ iff $e \in [e_1, e_2]$;
- if $e_1(\Omega) \neq e_2(\Omega)$ then every open trajectory fills some g-1 rk-2 component $\mathcal{N}_i$ of $\mathcal{M}_e$.

All $\{\hat{\mathcal{N}}_i\}$ share the same hom class $l = [\hat{\mathcal{N}}_i] \in H_2(\mathbb{T}^3, \mathbb{Z})$

l is the same for all $e \in [e_1(\Omega), e_2(\Omega)]$

and it invariant for small perturbations of Ω



4 – The topological invariant

Theorem [Dynnikov] Be H a generic real function on $\mathbb{T}^3$.
Then there exist two continous functions $e_{1,2} : \mathbb{R}P^2 \to \mathbb{R}$ s.t.

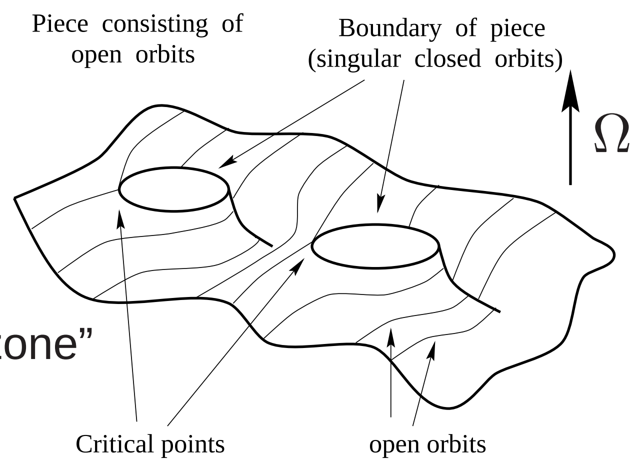
- $e_1 \leq e_2$;
- open trajectories arise on $\mathcal{M}_e = H^{-1}(e)$ iff $e \in [e_1, e_2]$;
- if $e_1(\Omega) \neq e_2(\Omega)$ then every open trajectory fills some g-1 rk-2 component $\mathcal{N}_i$ of $\mathcal{M}_e$.

All $\{\hat{\mathcal{N}}_i\}$ share the same hom class $l = [\hat{\mathcal{N}}_i] \in H_2(\mathbb{T}^3, \mathbb{Z})$

l is the same for all $e \in [e_1(\Omega), e_2(\Omega)]$

and it invariant for small perturbations of Ω

The set D_l of all Ω sharing a top inv l is called "stability zone"



4 – The topological invariant

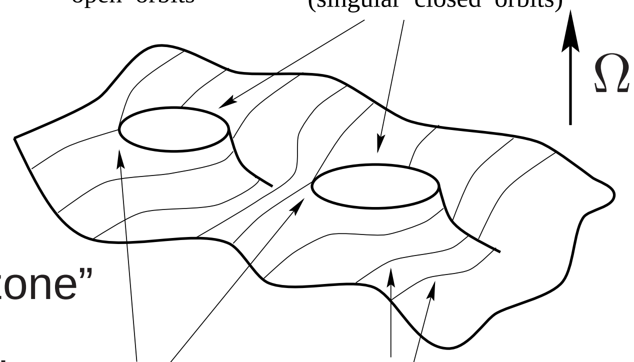
Theorem [Dynnikov] Be H a generic real function on $\mathbb{T}^3$.
Then there exist two continous functions $e_{1,2} : \mathbb{R}P^2 \to \mathbb{R}$ s.t.

- $e_1 \leq e_2$;
- open trajectories arise on $\mathcal{M}_e = H^{-1}(e)$ iff $e \in [e_1, e_2]$;
- if $e_1(\Omega) \neq e_2(\Omega)$ then every open trajectory fills some g-1 rk-2 component $\mathcal{N}_i$ of $\mathcal{M}_e$.

All $\{\hat{\mathcal{N}}_i\}$ share the same hom class $l = [\hat{\mathcal{N}}_i] \in H_2(\mathbb{T}^3, \mathbb{Z})$

l is the same for all $e \in [e_1(\Omega), e_2(\Omega)]$

and it invariant for small perturbations of Ω



The set D_l of all Ω sharing a top inv l is called "stability zone"

The inv. l is enough to describe the aymptotics of open orbits:

if $\Omega \in D_l$ then all open obits are str. asympt. to $\Omega \times l$.

5 – The Stereographic Map

Theorem [Zorich,Dynnikov,DL]

Be SM$(\mathcal{M}_e)$ the set of stability zones (stereographic map) at a generic energy e:

5 – The Stereographic Map

Theorem [Zorich,Dynnikov,DL]

Be SM$(\mathcal{M}_e)$ the set of stability zones (stereographic map) at a generic energy e:

- SM$(\mathcal{M}_e)$ is the disjoint union of finitely many open sets;

5 – The Stereographic Map

Theorem [Zorich,Dynnikov,DL]

Be SM$(\mathcal{M}_e)$ the set of stability zones (stereographic map) at a generic energy e:

- SM$(\mathcal{M}_e)$ is the disjoint union of finitely many open sets;
- the set of Ω who induce open orbits whose closure has genus > 1 is 0.

5 – The Stereographic Map

Theorem [Zorich,Dynnikov,DL]

Be SM$(\mathcal{M}_e)$ the set of stability zones (stereographic map) at a generic energy e:

- SM$(\mathcal{M}_e)$ is the disjoint union of finitely many open sets;
- the set of Ω who induce open orbits whose closure has genus > 1 is 0.

Be SM$(H) = \bigcup$ SM$(\mathcal{M}_e)$ the SM relative to the function H:

5 – The Stereographic Map

Theorem [Zorich,Dynnikov,DL]

Be SM$(\mathcal{M}_e)$ the set of stability zones (stereographic map) at a generic energy e:

- SM$(\mathcal{M}_e)$ is the disjoint union of finitely many open sets;
- the set of Ω who induce open orbits whose closure has genus > 1 is 0.

Be SM$(H) = \bigcup$ SM$(\mathcal{M}_e)$ the SM relative to the function H:

- either there is only 1 invariant l, and SM$(H) = \mathbb{R}P^2$, or there are countably many and their boundaries touch only in countably many points in a "fractal-like" way;

5 – The Stereographic Map

Theorem [Zorich,Dynnikov,DL]

Be SM$(\mathcal{M}_e)$ the set of stability zones (stereographic map) at a generic energy e:

- SM$(\mathcal{M}_e)$ is the disjoint union of finitely many open sets;
- the set of Ω who induce open orbits whose closure has genus > 1 is 0.

Be SM$(H) = \bigcup$SM$(\mathcal{M}_e)$ the SM relative to the function H:

- either there is only 1 invariant l, and SM$(H) = \mathbb{R}P^2$, or there are countably many and their boundaries touch only in countably many points in a "fractal-like" way;
- Ω rational $\Longrightarrow \Omega \in$ SM(H), in particular $\overline{\text{SM}(H)} = \mathbb{R}P^2$;

5 – The Stereographic Map

Theorem [Zorich,Dynnikov,DL]

Be SM$(\mathcal{M}_e)$ the set of stability zones (stereographic map) at a generic energy e:

- SM$(\mathcal{M}_e)$ is the disjoint union of finitely many open sets;
- the set of Ω who induce open orbits whose closure has genus > 1 is 0.

Be SM$(H) = \bigcup$SM$(\mathcal{M}_e)$ the SM relative to the function H:

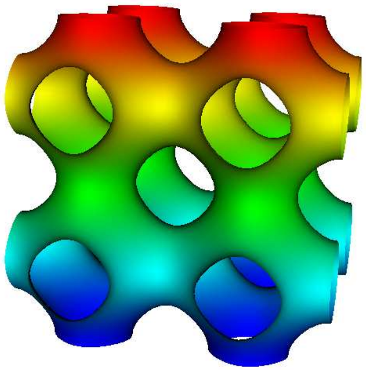
- either there is only 1 invariant l, and SM$(H) = \mathbb{R}P^2$, or there are countably many and their boundaries touch only in countably many points in a "fractal-like" way;
- Ω rational $\Longrightarrow \Omega \in$ SM(H), in particular $\overline{\text{SM}(H)} = \mathbb{R}P^2$;
- if there is more than one label then there exist uncountably many "ergodic direction" and the closure of the set of labels is equal to the set of zones boundaries plus the set of ergodic directions.

6 – The Schwarz's P-surface

$$\cos(x)+\cos(y)+\cos(z)=0$$

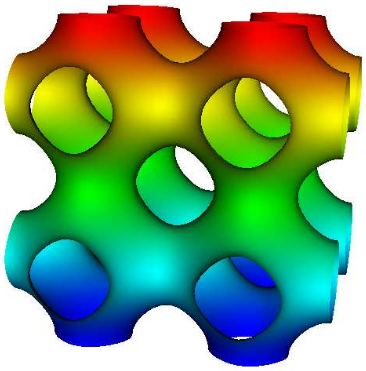
6 – The Schwarz's P-surface

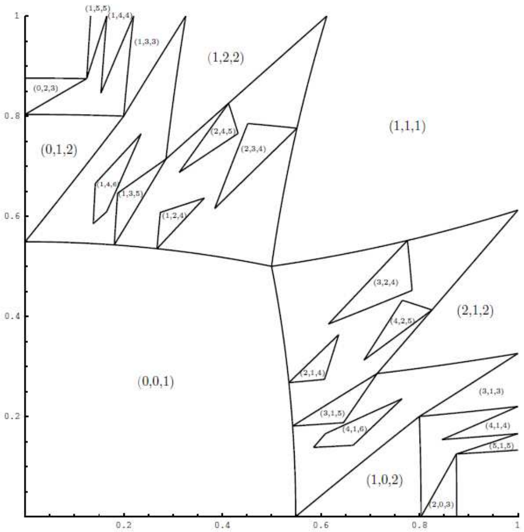
$$\cos(x)+\cos(y)+\cos(z)=0$$



6 – The Schwarz's P-surface

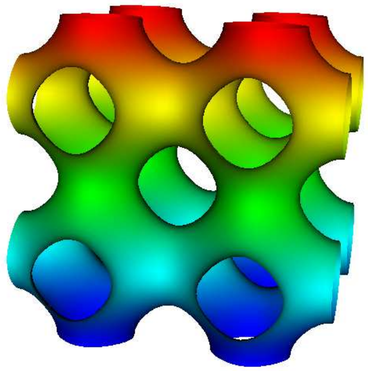
$$\cos(x) + \cos(y) + \cos(z) = 0$$

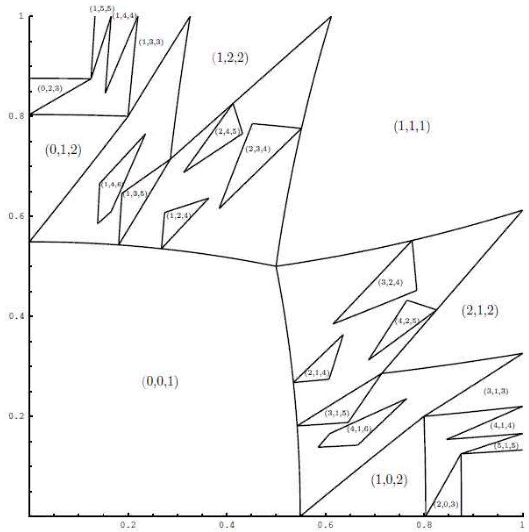


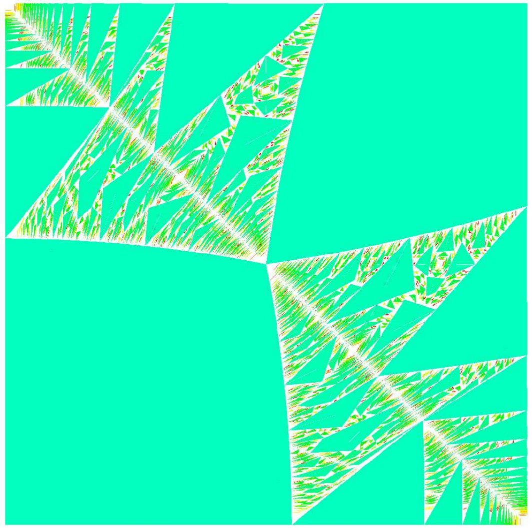


6 – The Schwarz's P-surface

$$\cos(x) + \cos(y) + \cos(z) = 0$$

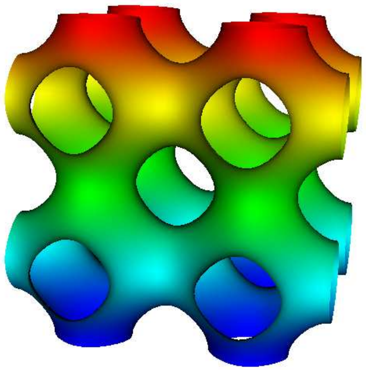


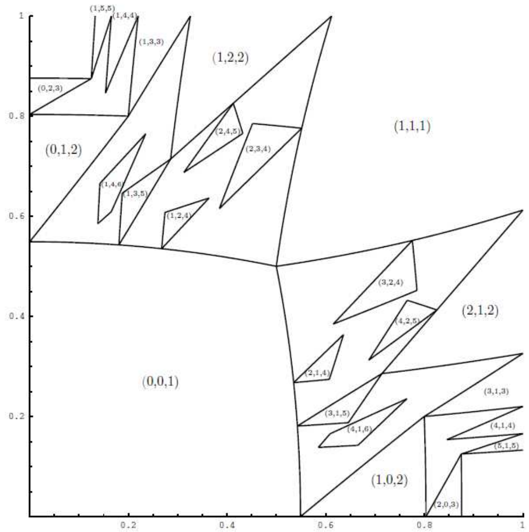


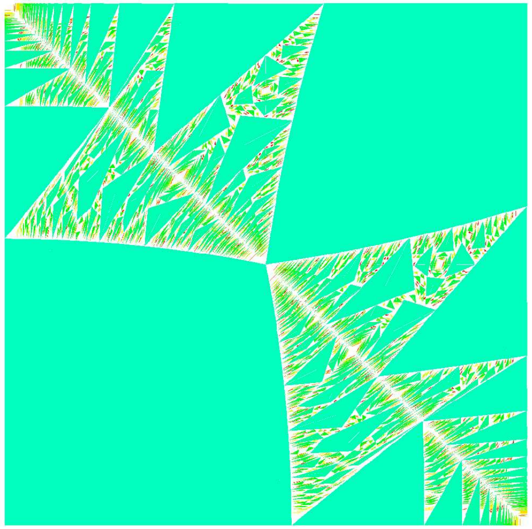


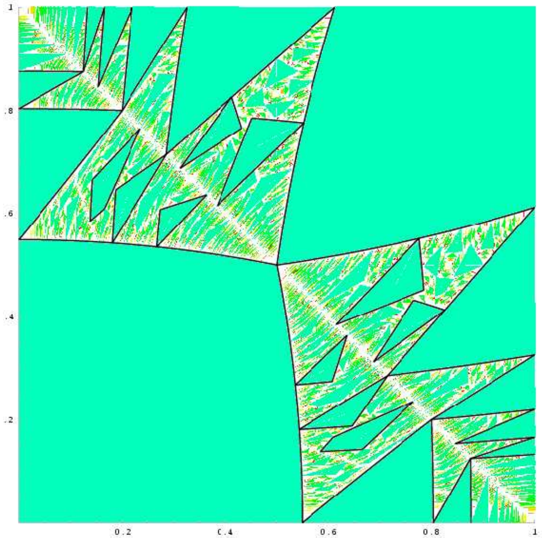
6 – The Schwarz’s P-surface

$$\cos(x) + \cos(y) + \cos(z) = 0$$



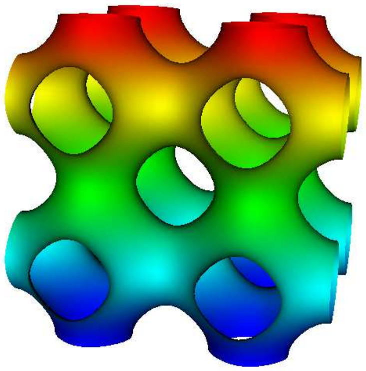


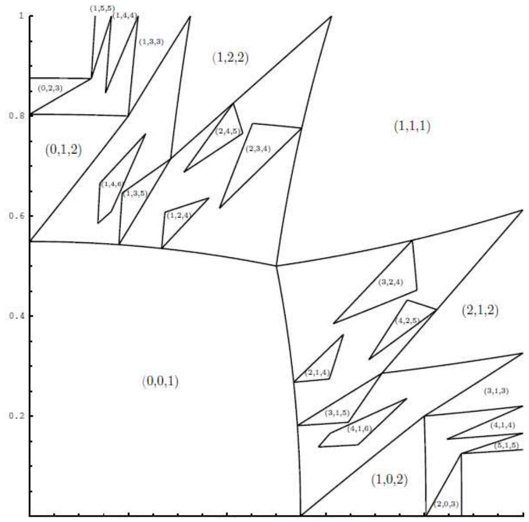


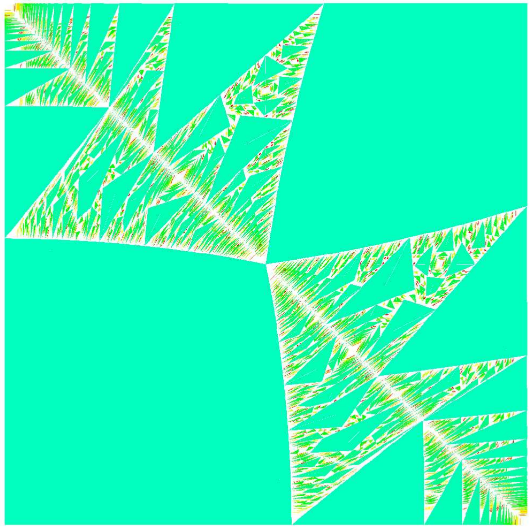


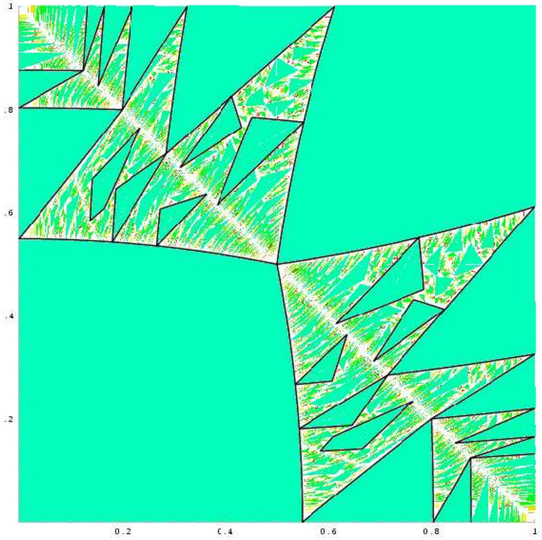
6 – The Schwarz's P-surface

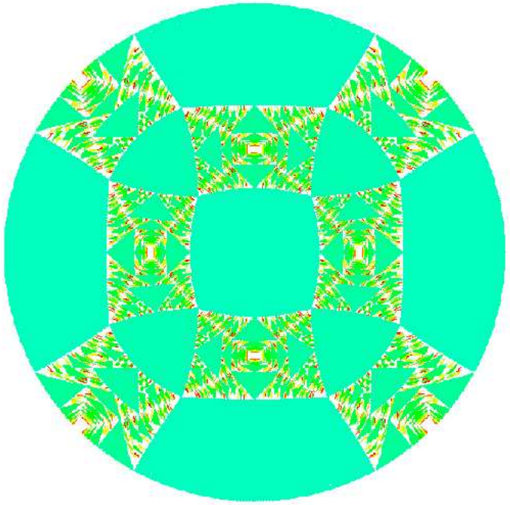
$$\cos(x)+\cos(y)+\cos(z)=0$$







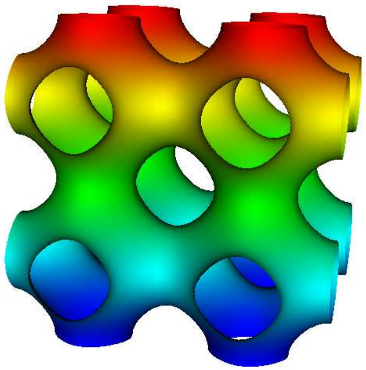


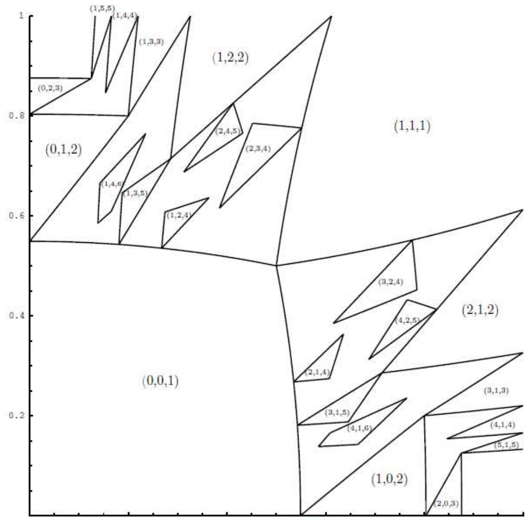


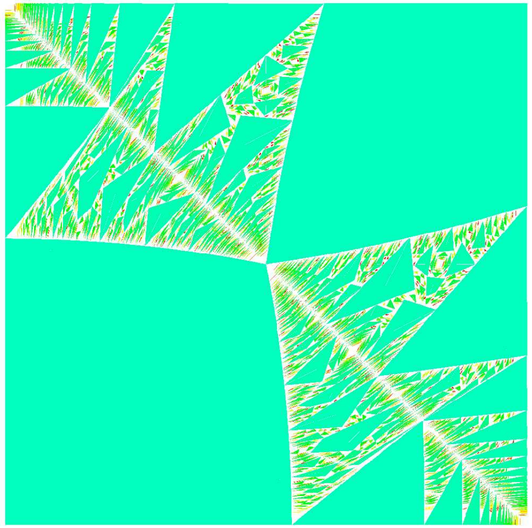
I.A. Dynnikov,R. De Leo

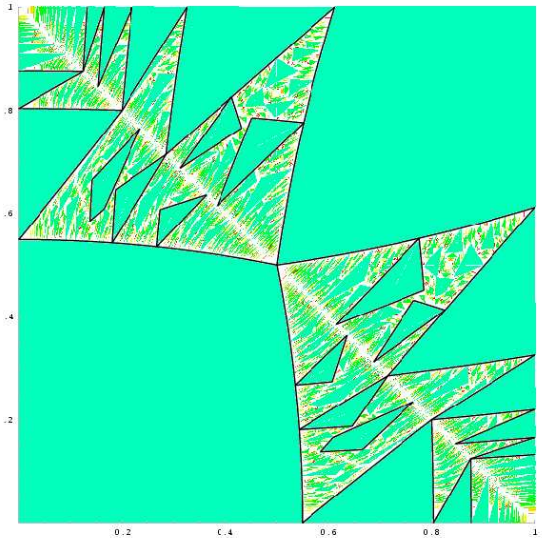
6 – The Schwarz's P-surface

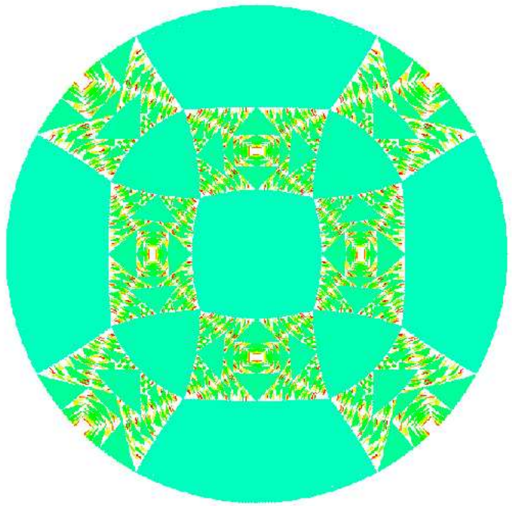
$$\cos(x) + \cos(y) + \cos(z) = 0$$

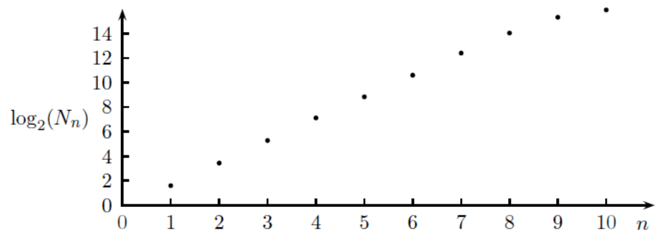








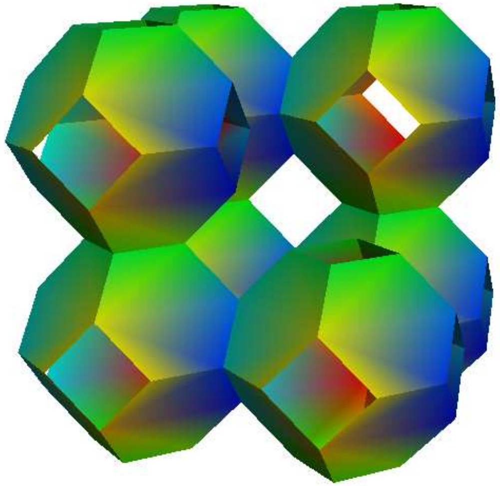




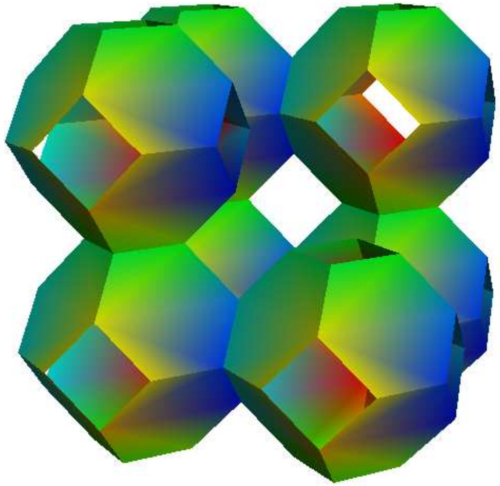
$$d_{box} \simeq 1.77$$

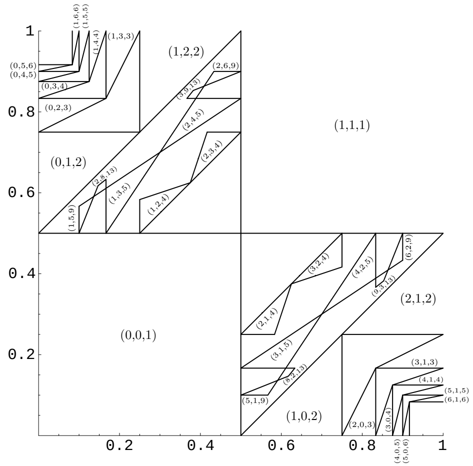
7 – The regular skew polyhedron $\{6, 4 \,|\, 4\}$

7 – The regular skew polyhedron $\{6, 4 \,|\, 4\}$

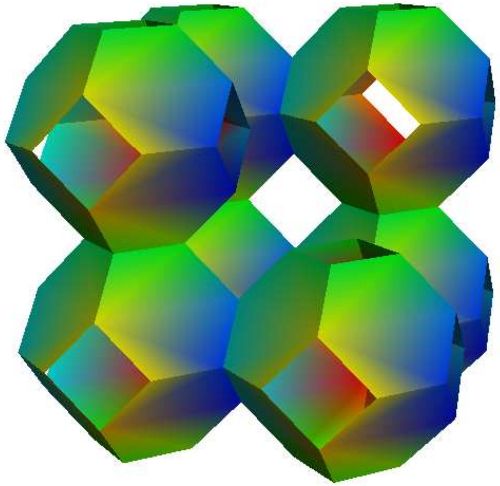


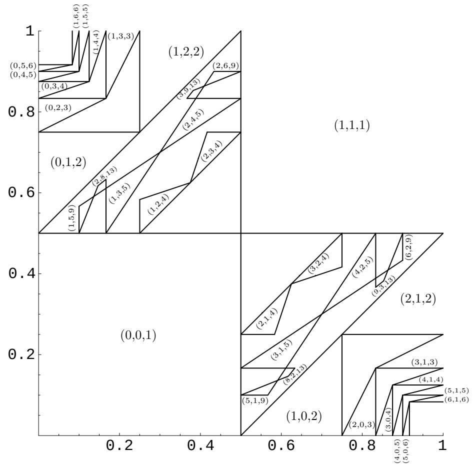
7 – The regular skew polyhedron $\{6,4\,|\,4\}$

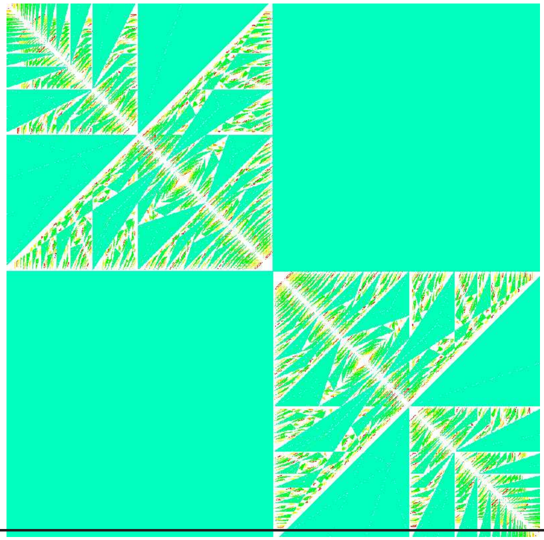




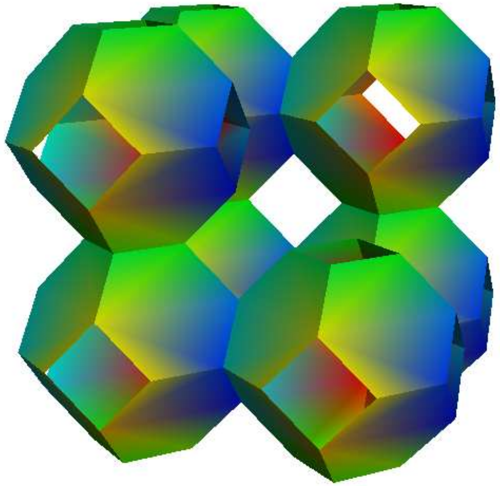
7 – The regular skew polyhedron $\{6, 4\,|\,4\}$

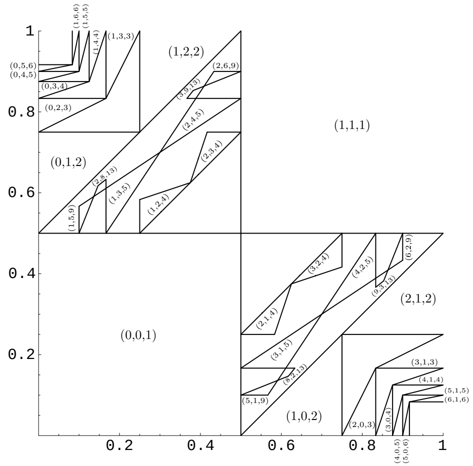


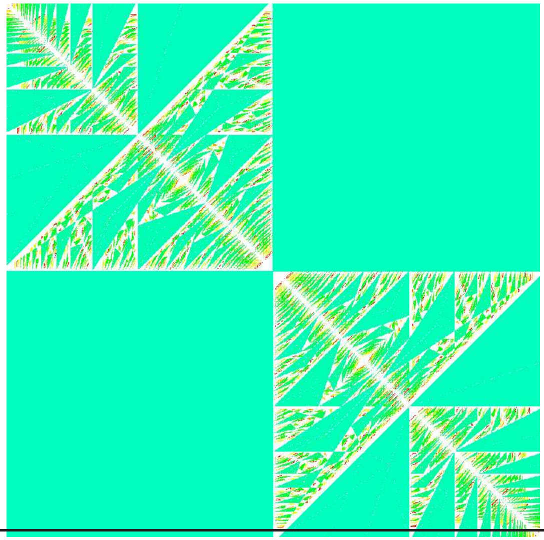


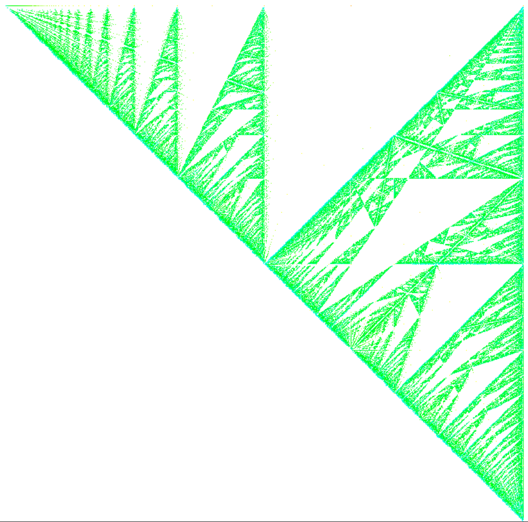


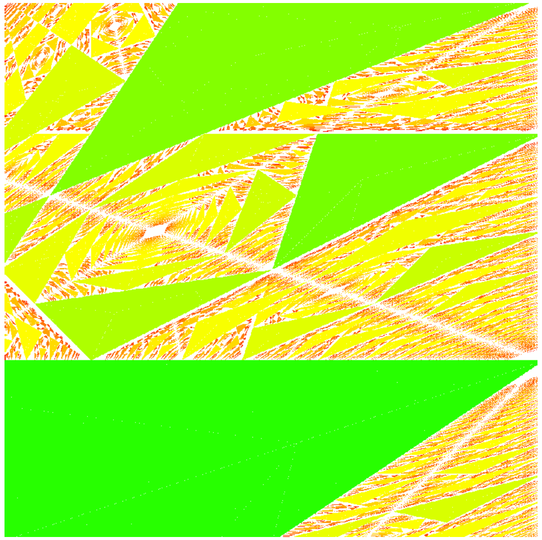
7 – The regular skew polyhedron $\{6, 4\,|\,4\}$



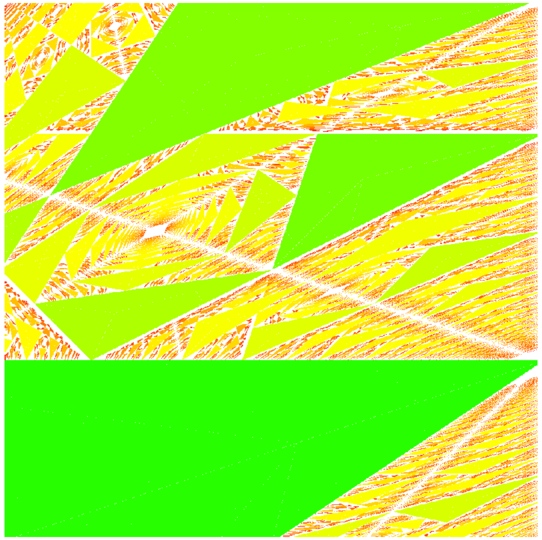


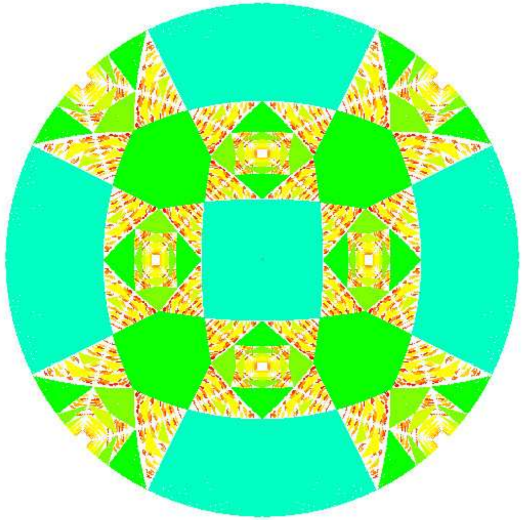


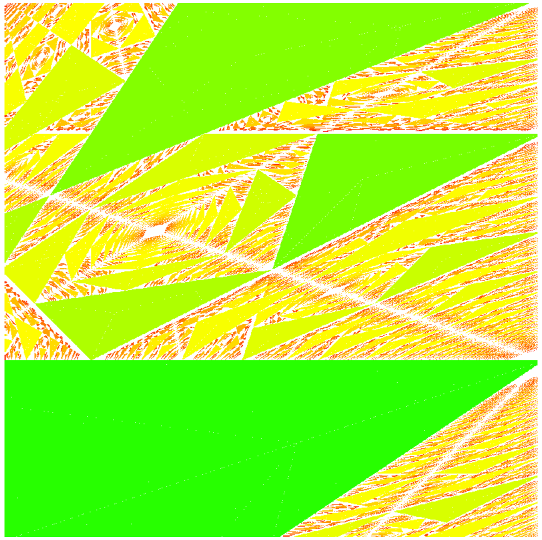


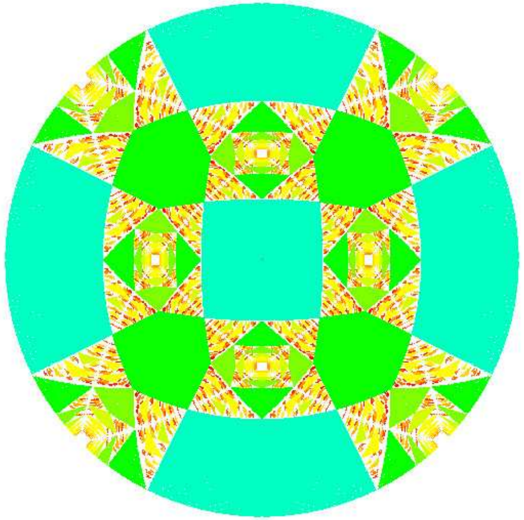


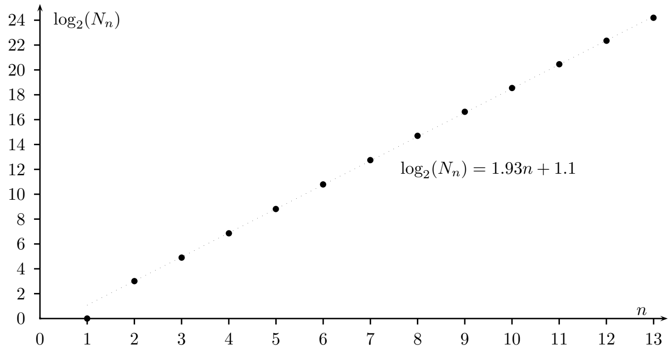
Plane sections of $\{4,6\,|\,4\}$





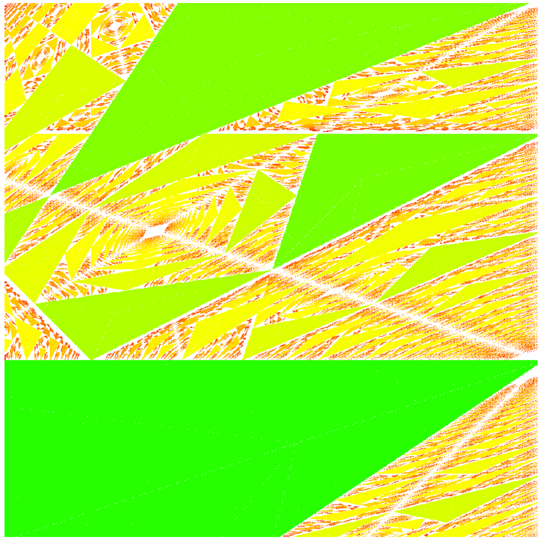


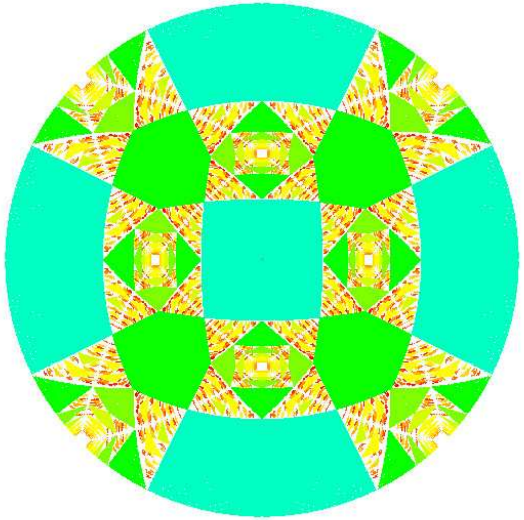


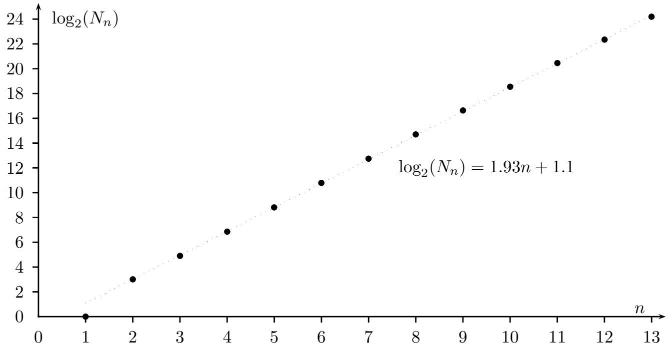


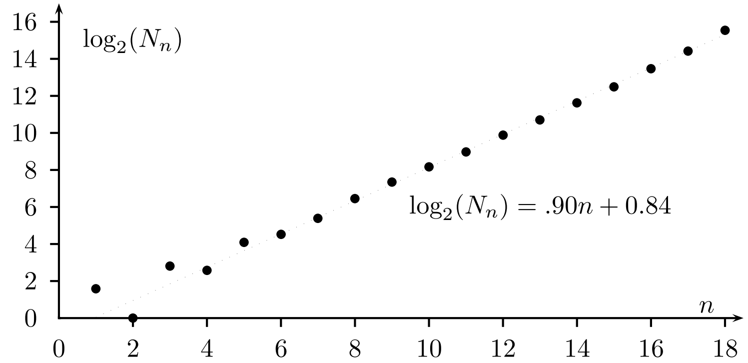
$\log_2(N_n)$
$\log_2(N_n) = 1.93n + 1.1$
n
24
22
20
18
16
14
12
10
8
6
4
2
0
0 1 2 3 4 5 6 7 8 9 10 11 12 13

Plane sections of $\{4,6\,|\,4\}$







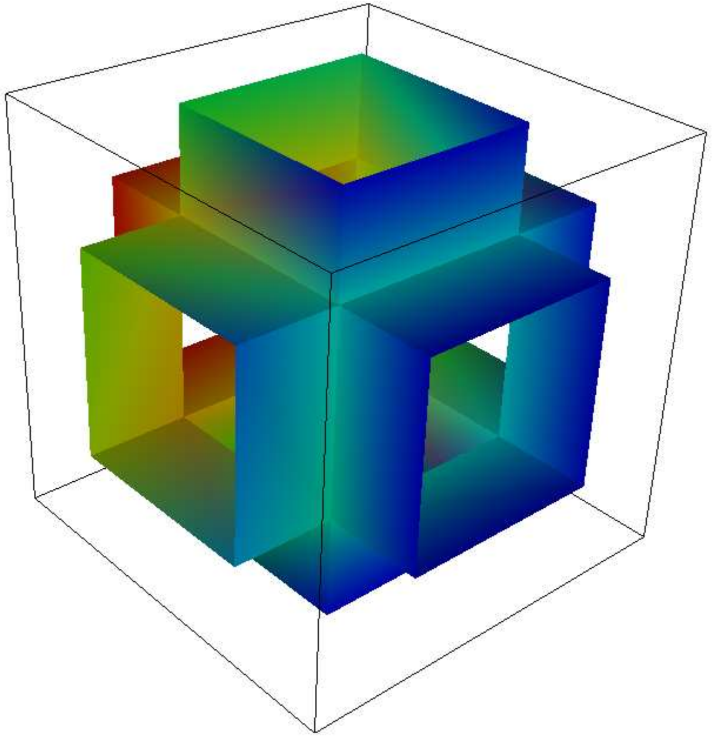


8 – Conjectures

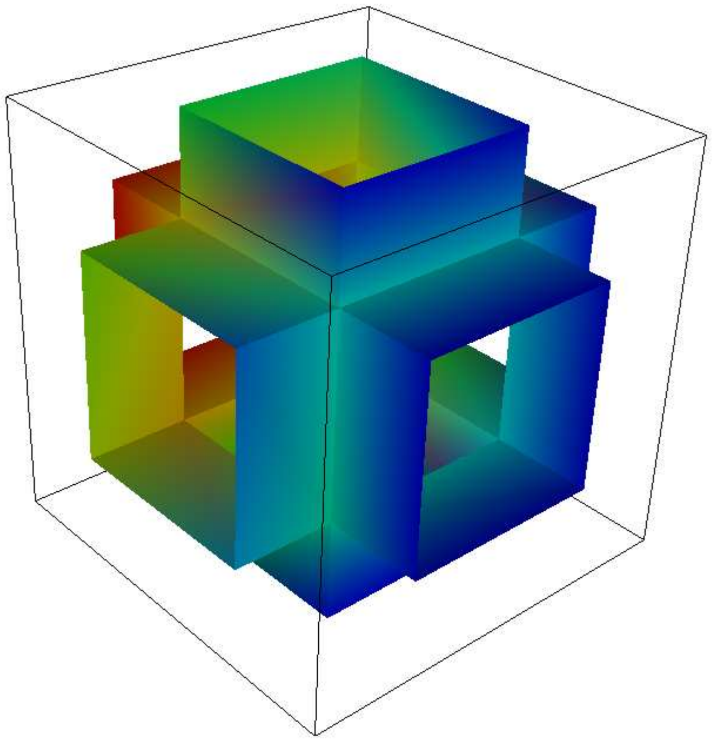
- The measure of the set of ergodic directions for a generic Hamiltonian is zero.
- The fractal dimension of the set of ergodic directions for a generic Hamiltonian is strictly between 1 and 2.
- The size of stability zones is bounded by $\frac{C}{||l||^3}$ for some constant C.

9 – The regular skew polyhedron $\{4,6\,|\,4\}$

9 – The regular skew polyhedron $\{4,6\,|\,4\}$

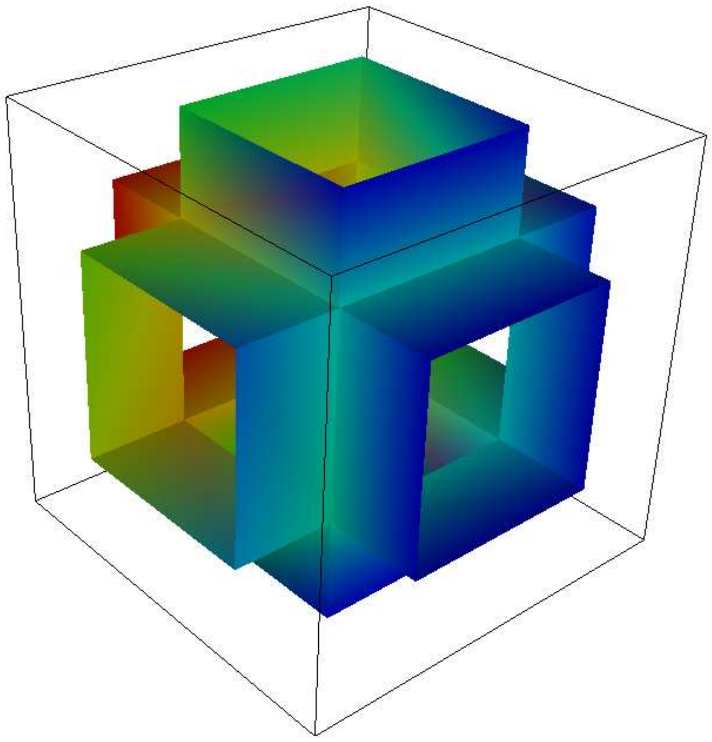


9 – The regular skew polyhedron $\{4,6\,|\,4\}$



Minimal discrete surface

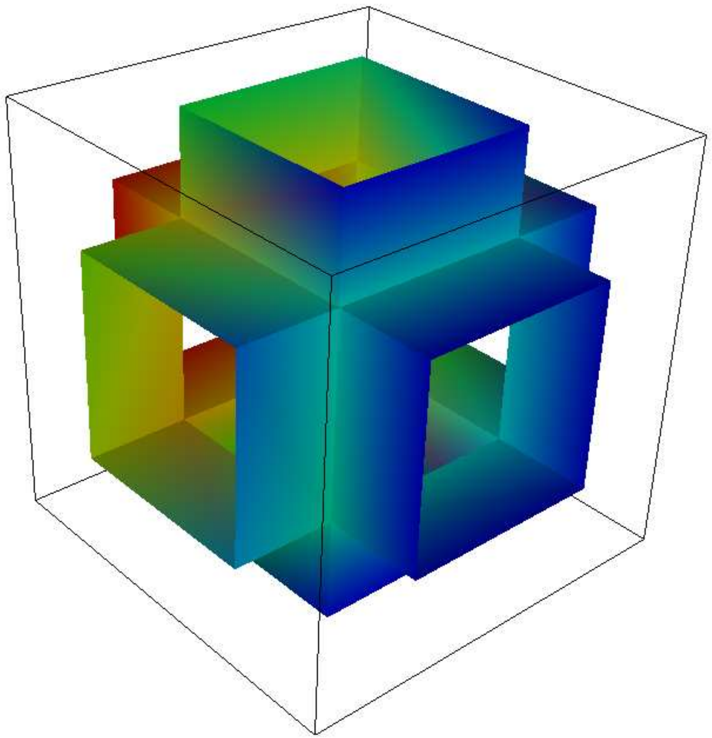
9 – The regular skew polyhedron $\{4,6\,|\,4\}$



Minimal discrete surface

It is one of the three regular skew polyhedra together with $\{6,4\,|\,4\}$ & $\{6,6\,|\,3\}$

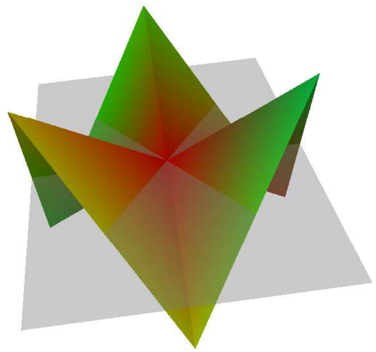
9 – The regular skew polyhedron $\{4,6\,|\,4\}$



Minimal discrete surface

It is one of the three regular skew polyhedra together with $\{6,4\,|\,4\}$ & $\{6,6\,|\,3\}$

All critical points of a generic Ω on it are of monkey saddle type



10 – A (new?) class of cut-out fractals

10 – A (new?) class of cut-out fractals

$a(l_a, m_a, n_a)$

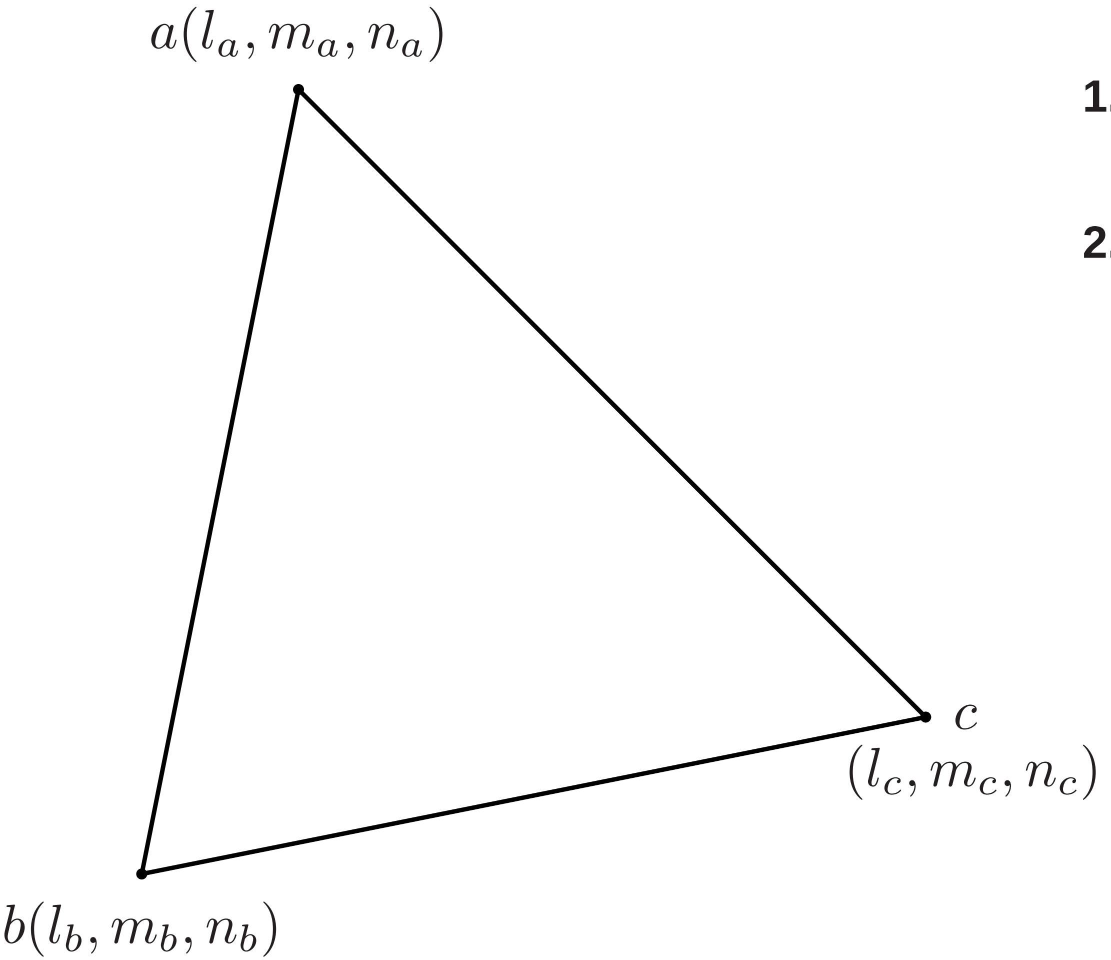
1. choose three rational directions in $\mathbb{R}P^2$

c

(l_c, m_c, n_c)

$b(l_b, m_b, n_b)$

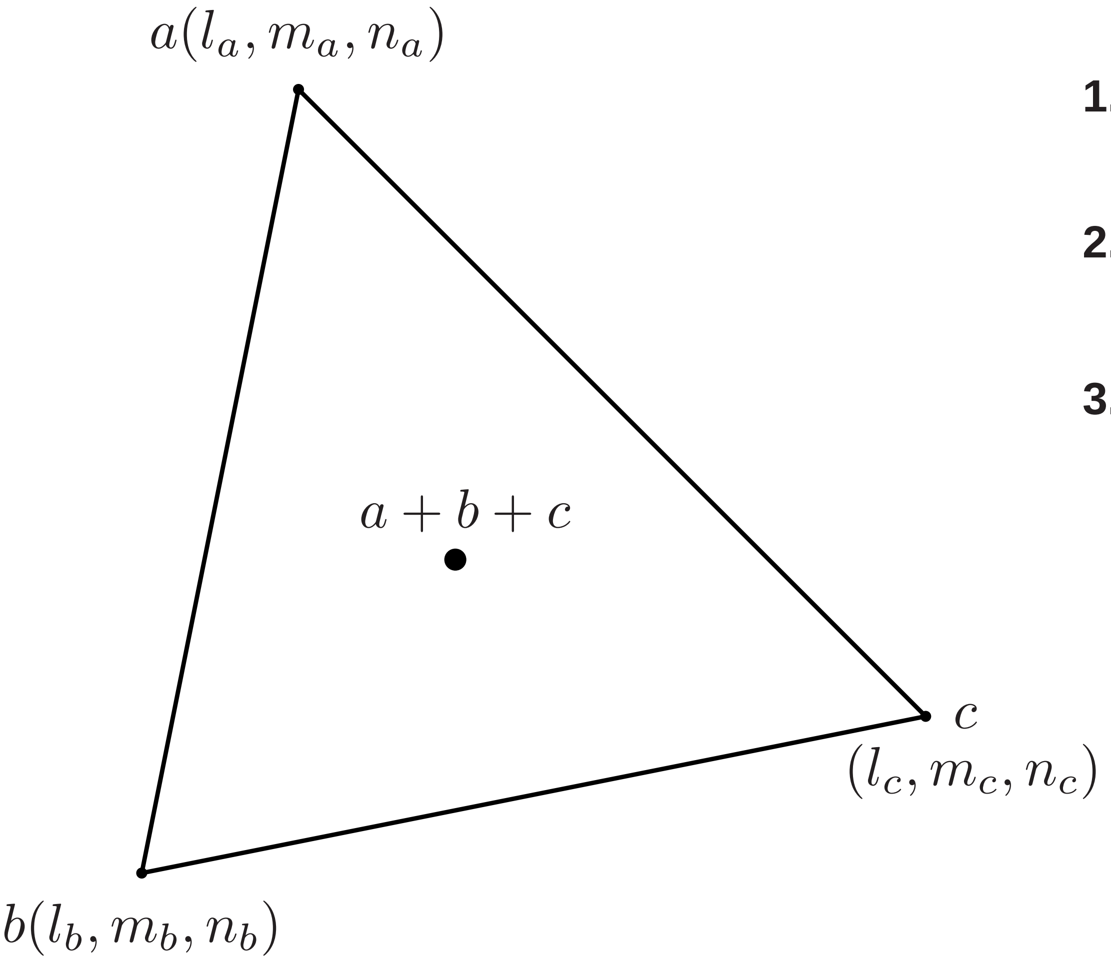
10 – A (new?) class of cut-out fractals



1. choose three rational directions in $\mathbb{R}P^2$

2. consider the triangle passing through them

10 – A (new?) class of cut-out fractals

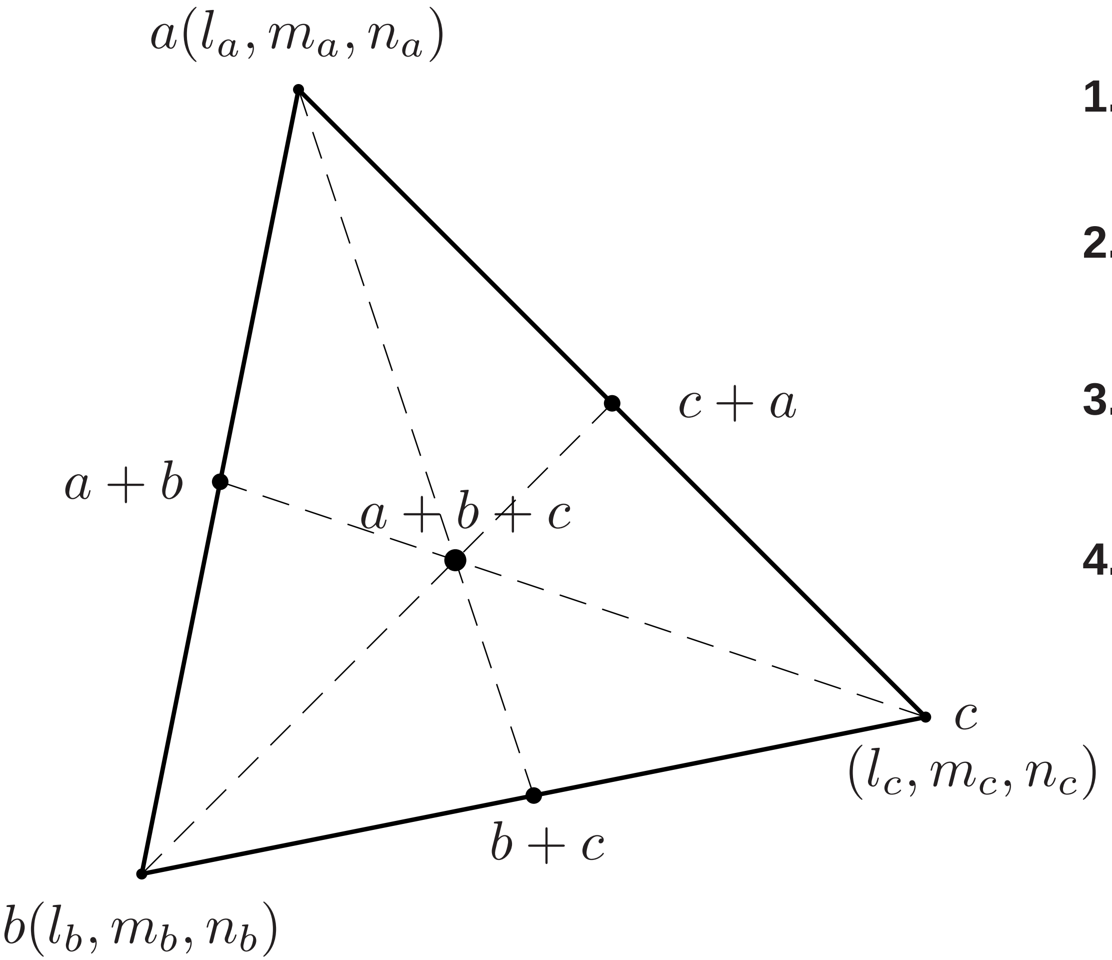


1. choose three rational directions in $\mathbb{R}P^2$

2. consider the triangle passing through them

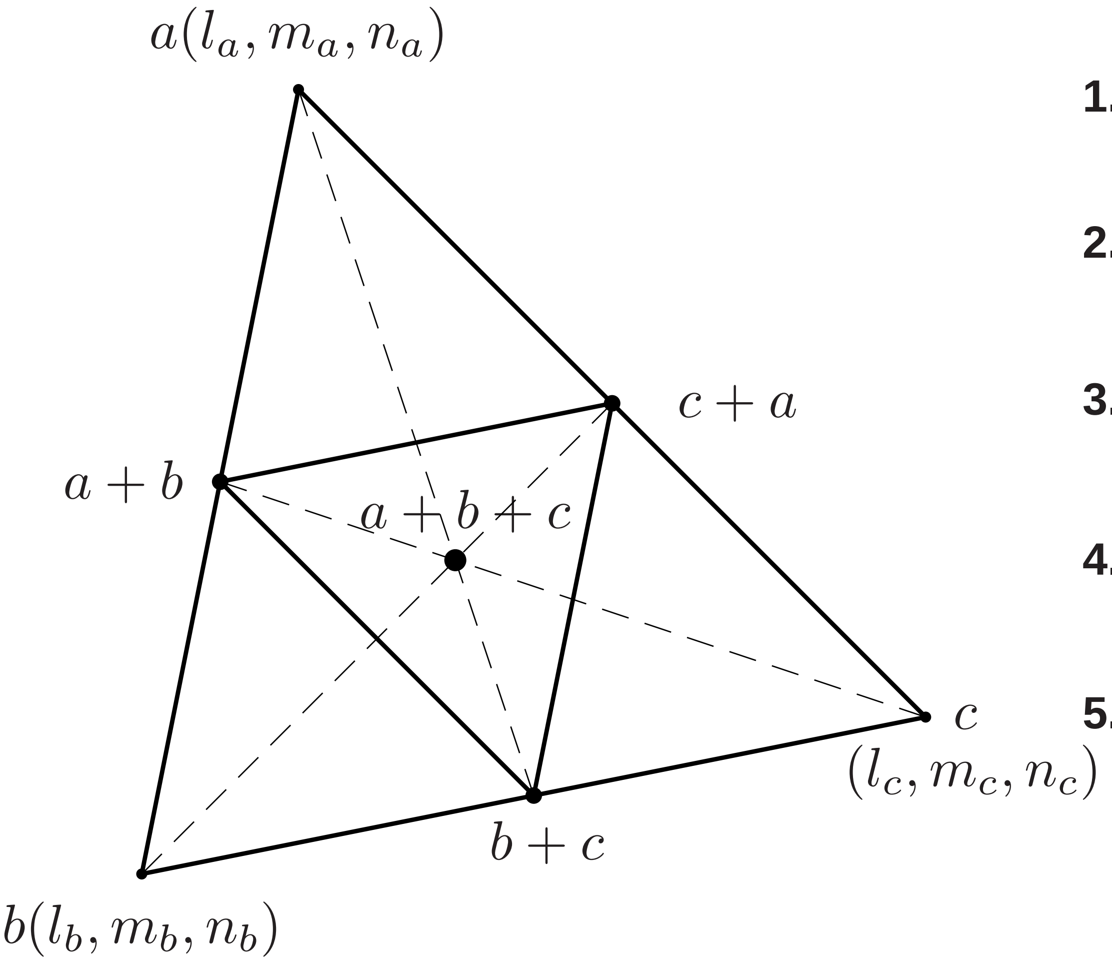
3. consider the point corresponding to the dir $a+b+c$

10 – A (new?) class of cut-out fractals



1. choose three rational directions in $\mathbb{R}P^2$
2. consider the triangle passing through them
3. consider the point corresponding to the dir $a+b+c$
4. project this point to the three sides from a, b & c

10 – A (new?) class of cut-out fractals



1. choose three rational directions in $\mathbb{R}P^2$

2. consider the triangle passing through them

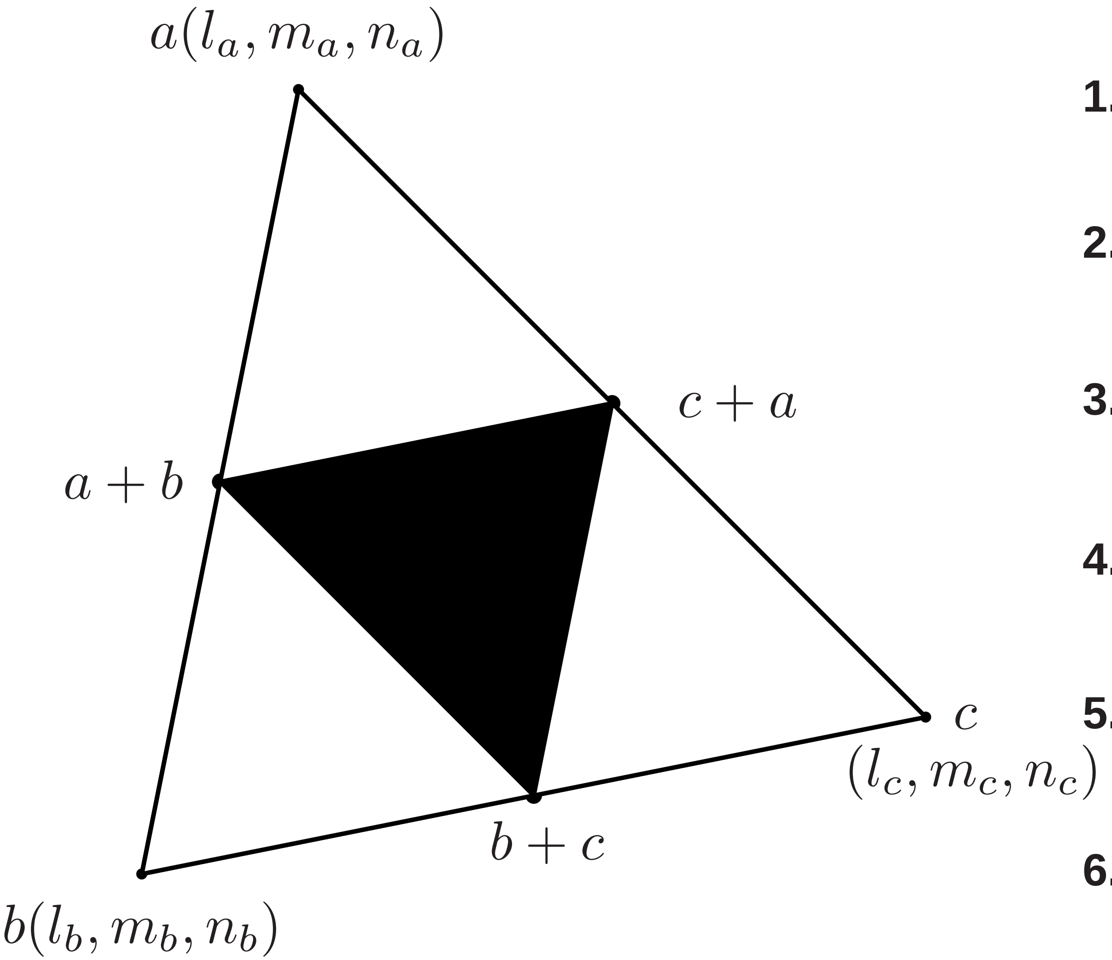
3. consider the point corresponding to the dir $a+b+c$

4. project this point to the three sides from a, b & c

5. consider the triangle with this proj. as vertices

I.A. Dynnikov,R. De Leo

10 – A (new?) class of cut-out fractals



1. choose three rational directions in $\mathbb{R}P^2$

2. consider the triangle passing through them

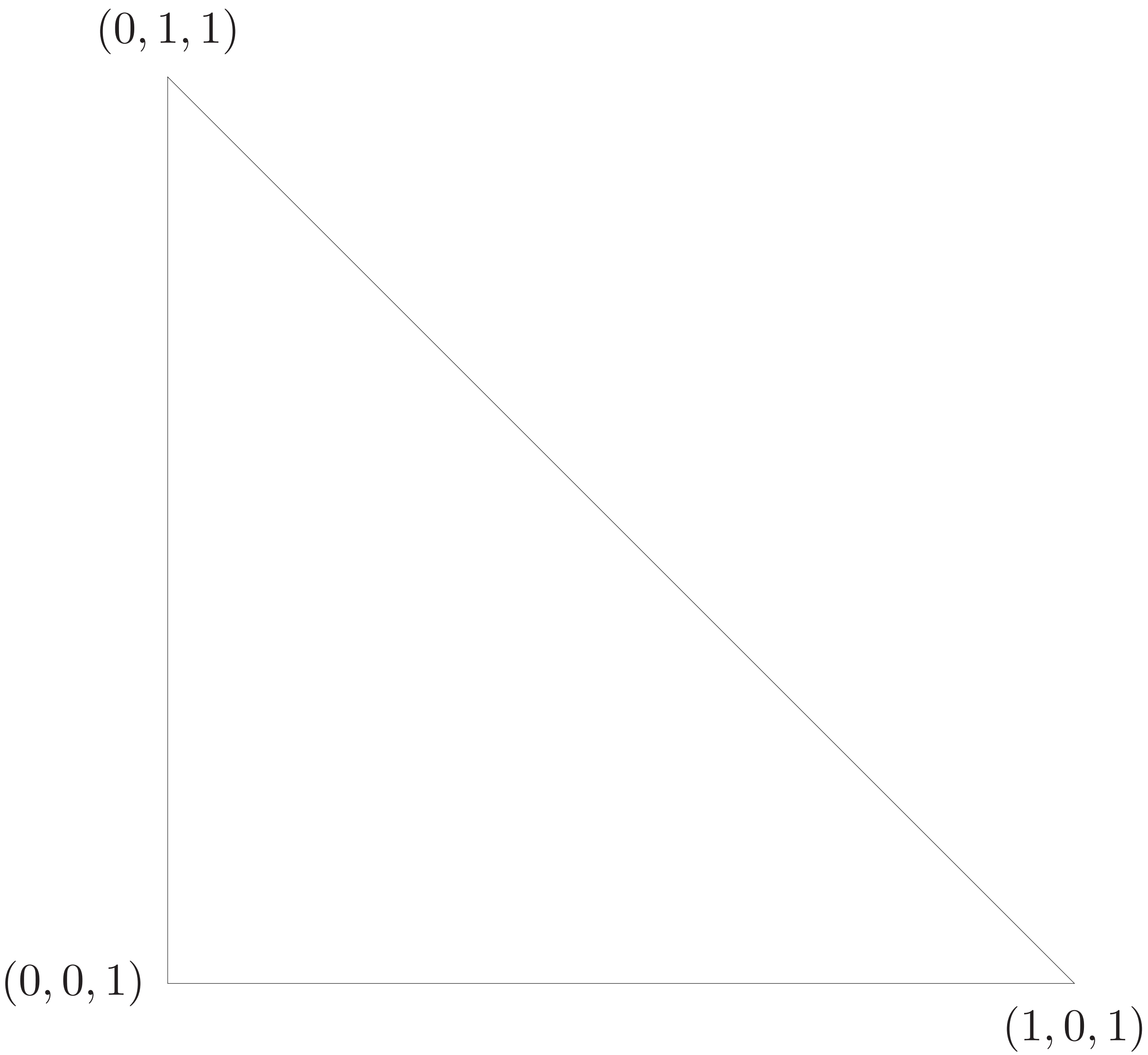
3. consider the point corresponding to the dir $a+b+c$

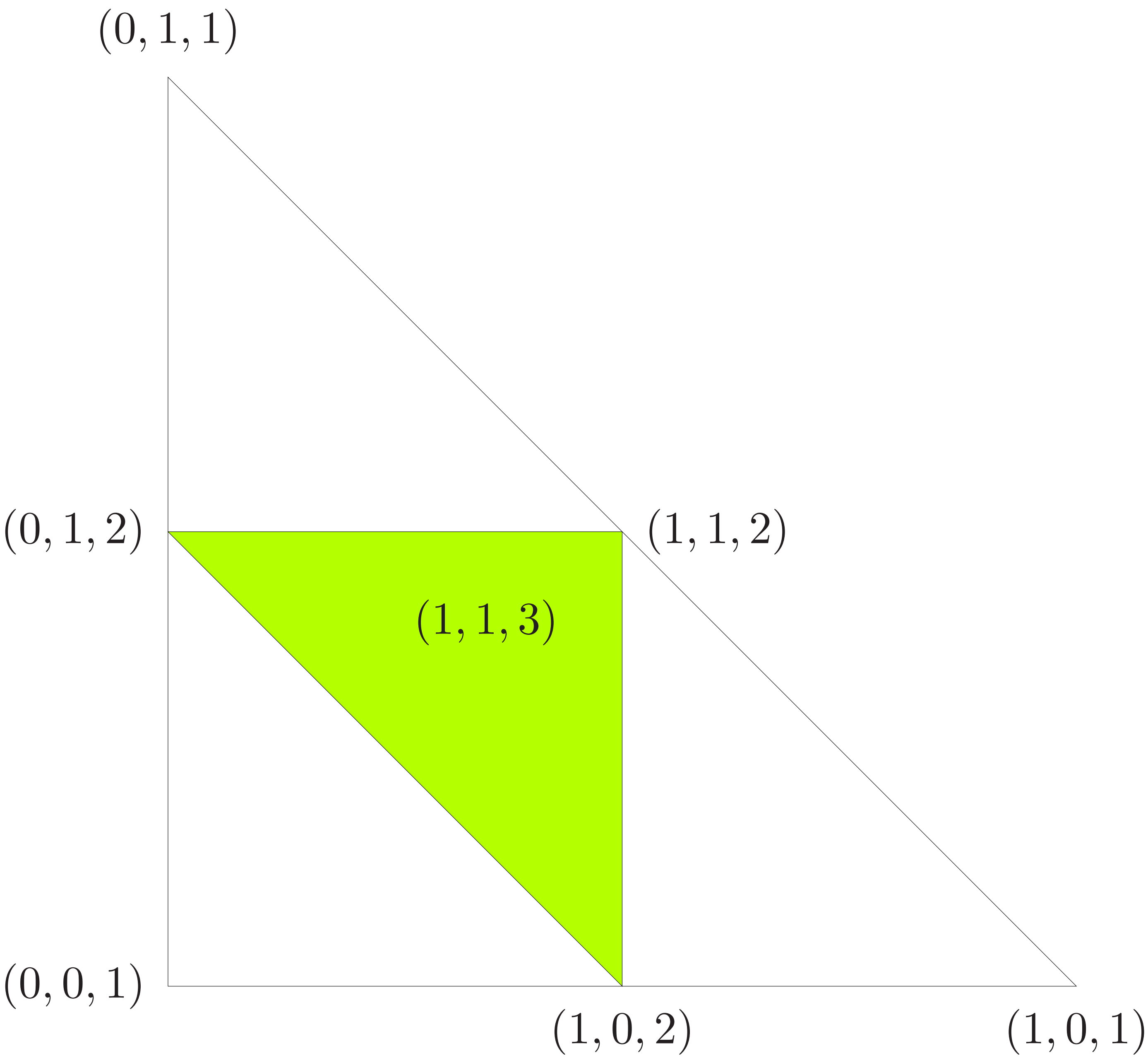
4. project this point to the three sides from a, b & c

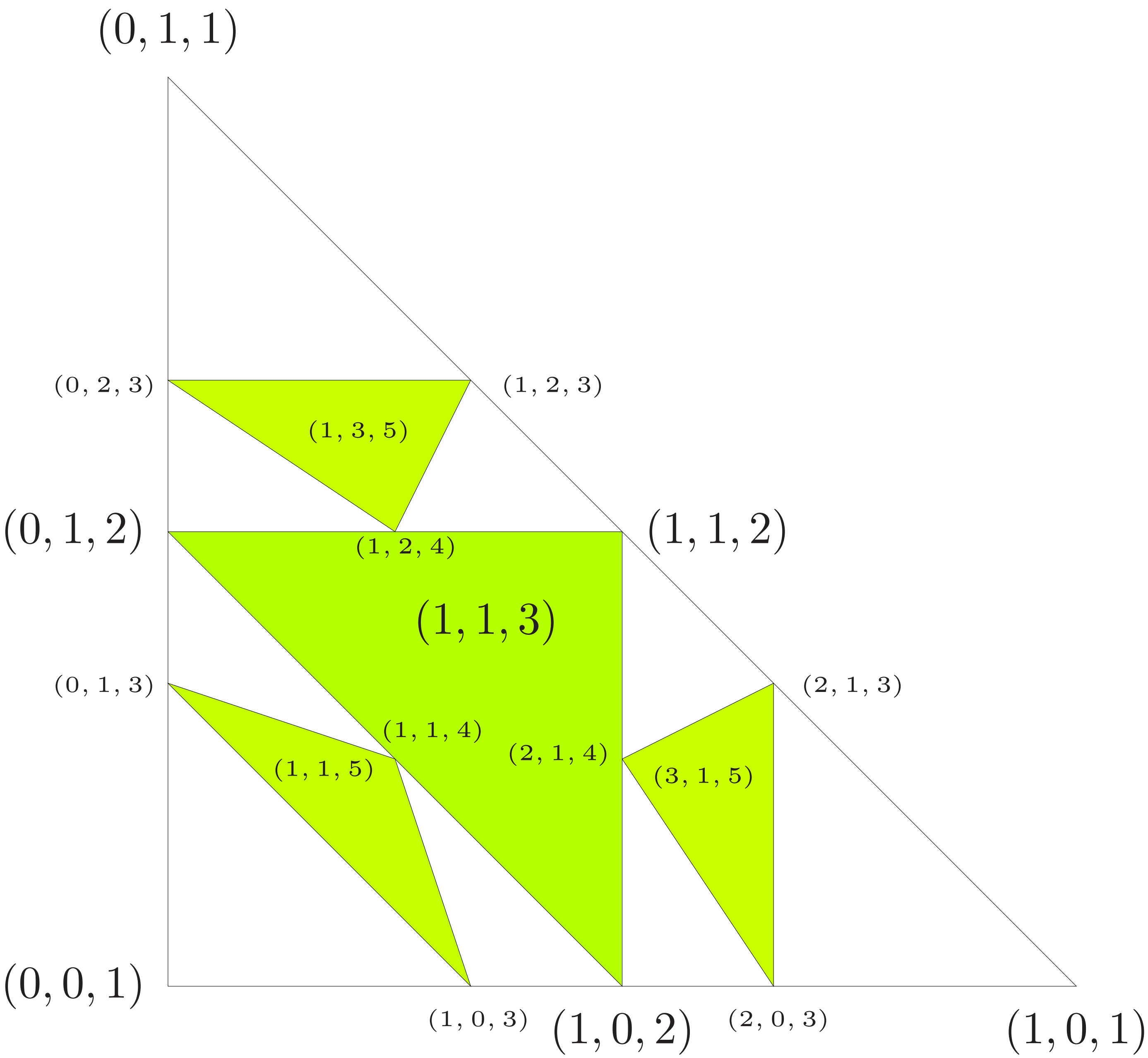
5. consider the triangle with this proj. as vertices

6. cut it out & repeat recursively

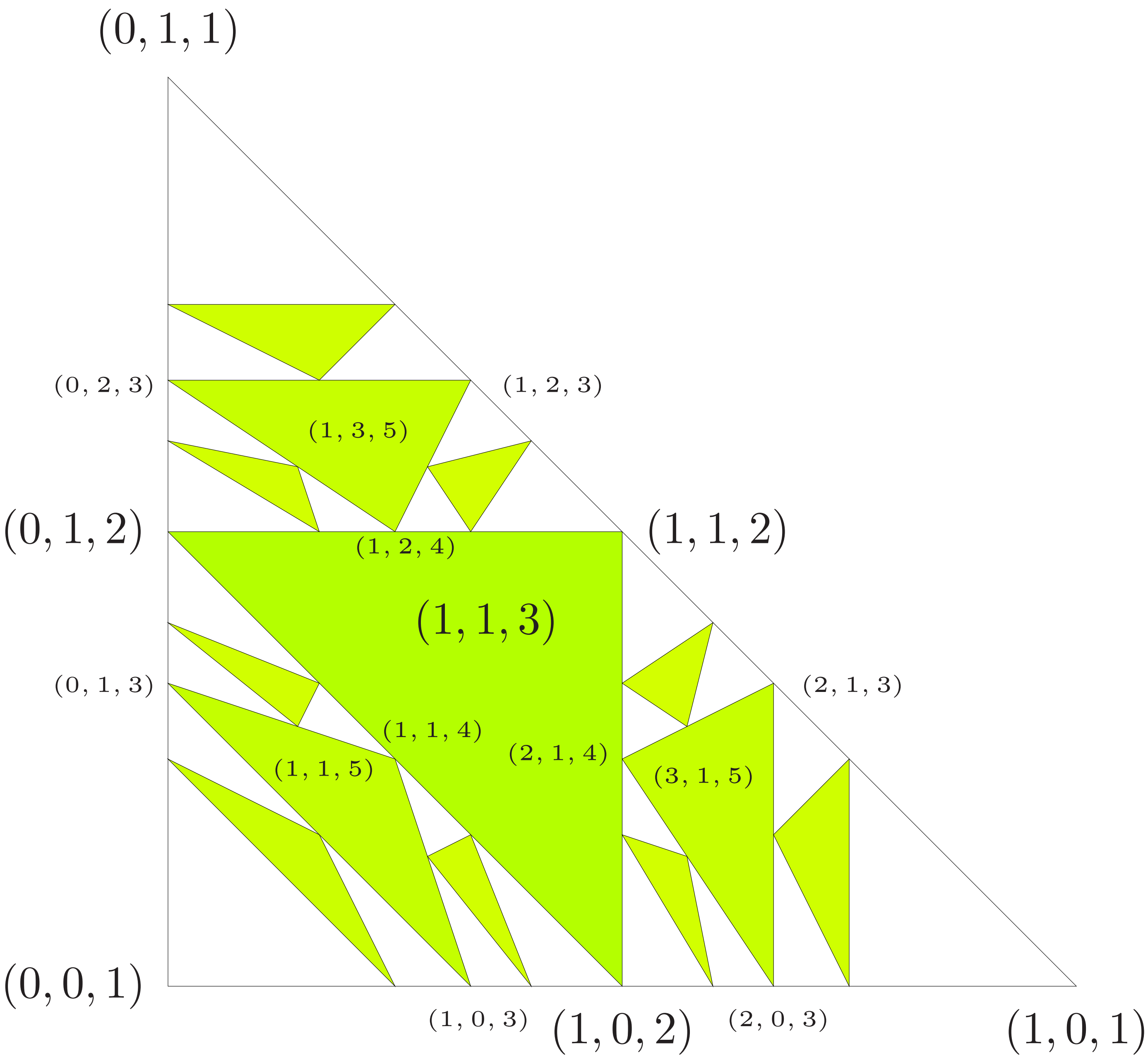
on the remaing triangles

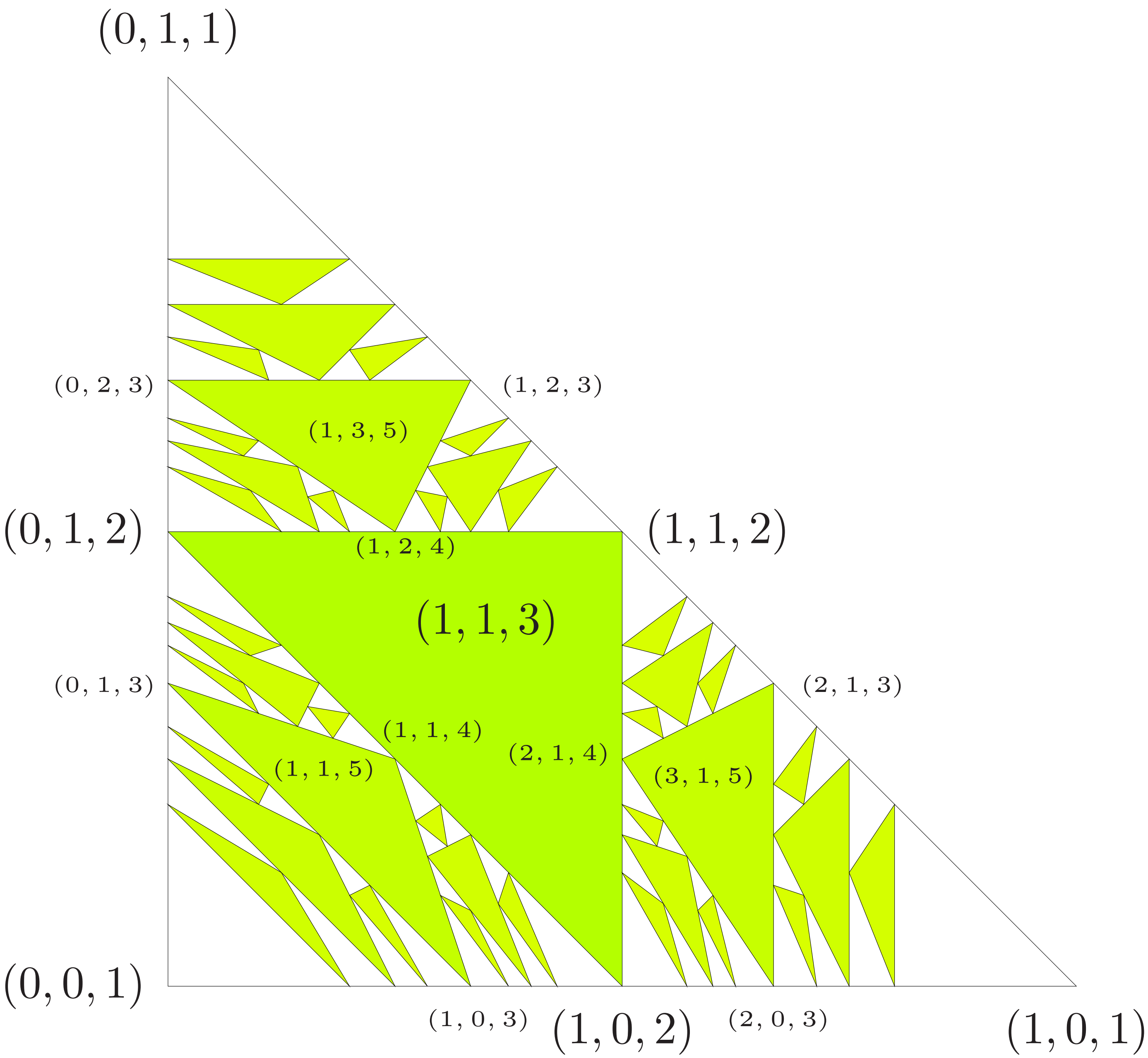

$(0,1,1)$
$(0,0,1)$
$(1,0,1)$


$(0,1,1)$
$(0,1,2)$
$(1,1,2)$
$(1,1,3)$
$(0,0,1)$
$(1,0,2)$
$(1,0,1)$

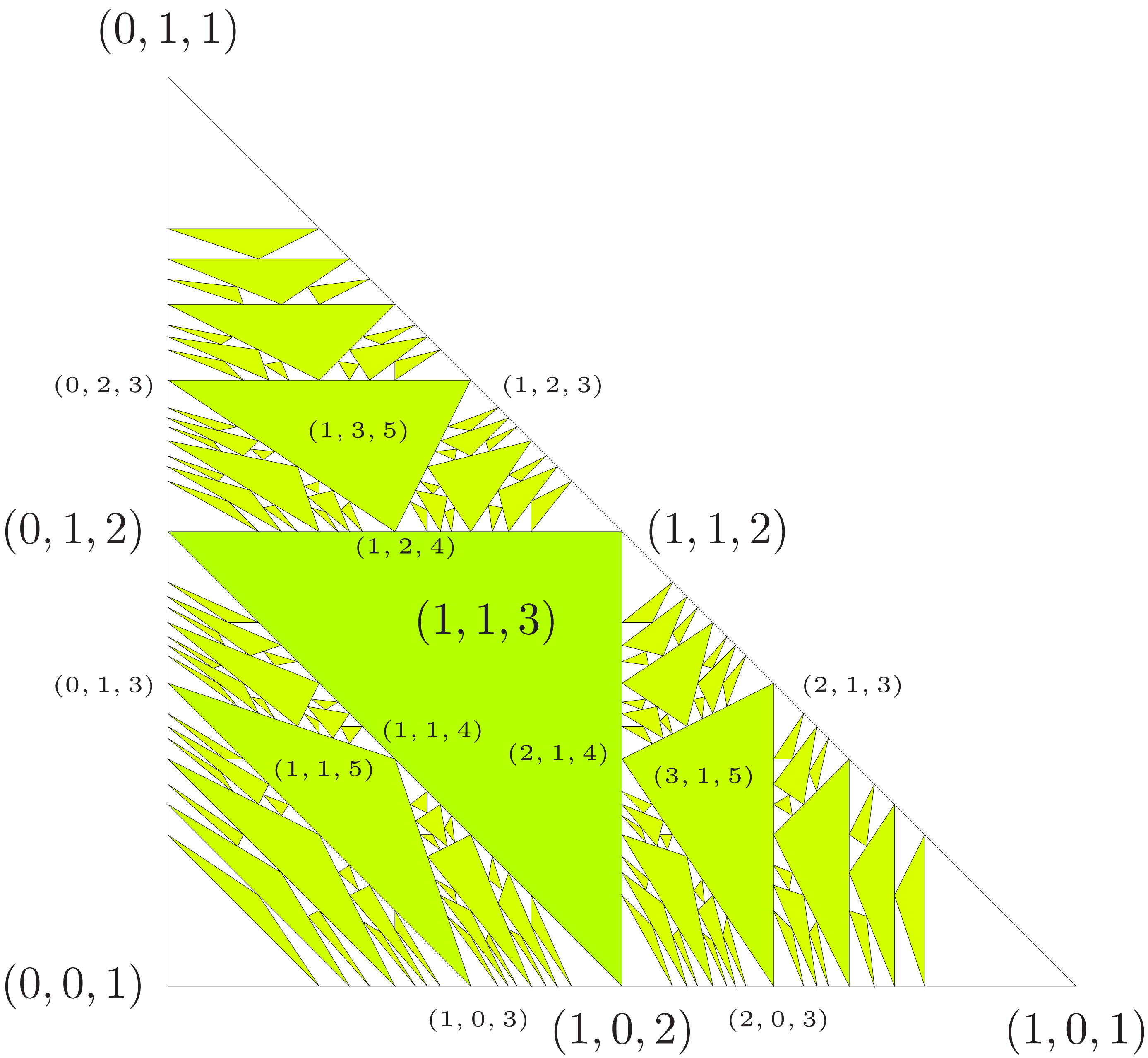


(0, 1, 1)
(0, 2, 3)
(1, 2, 3)
(1, 3, 5)
(0, 1, 2)
(1, 1, 2)
(1, 2, 4)
(1, 1, 3)
(0, 1, 3)
(2, 1, 3)
(1, 1, 4)
(2, 1, 4)
(1, 1, 5)
(3, 1, 5)
(0, 0, 1)
(1, 0, 3)
(1, 0, 2)
(2, 0, 3)
(1, 0, 1)


(0, 1, 1)
(0, 2, 3)
(1, 2, 3)
(1, 3, 5)
(0, 1, 2)
(1, 1, 2)
(1, 2, 4)
(1, 1, 3)
(0, 1, 3)
(2, 1, 3)
(1, 1, 4)
(2, 1, 4)
(1, 1, 5)
(3, 1, 5)
(0, 0, 1)
(1, 0, 3)
(1, 0, 2)
(2, 0, 3)
(1, 0, 1)

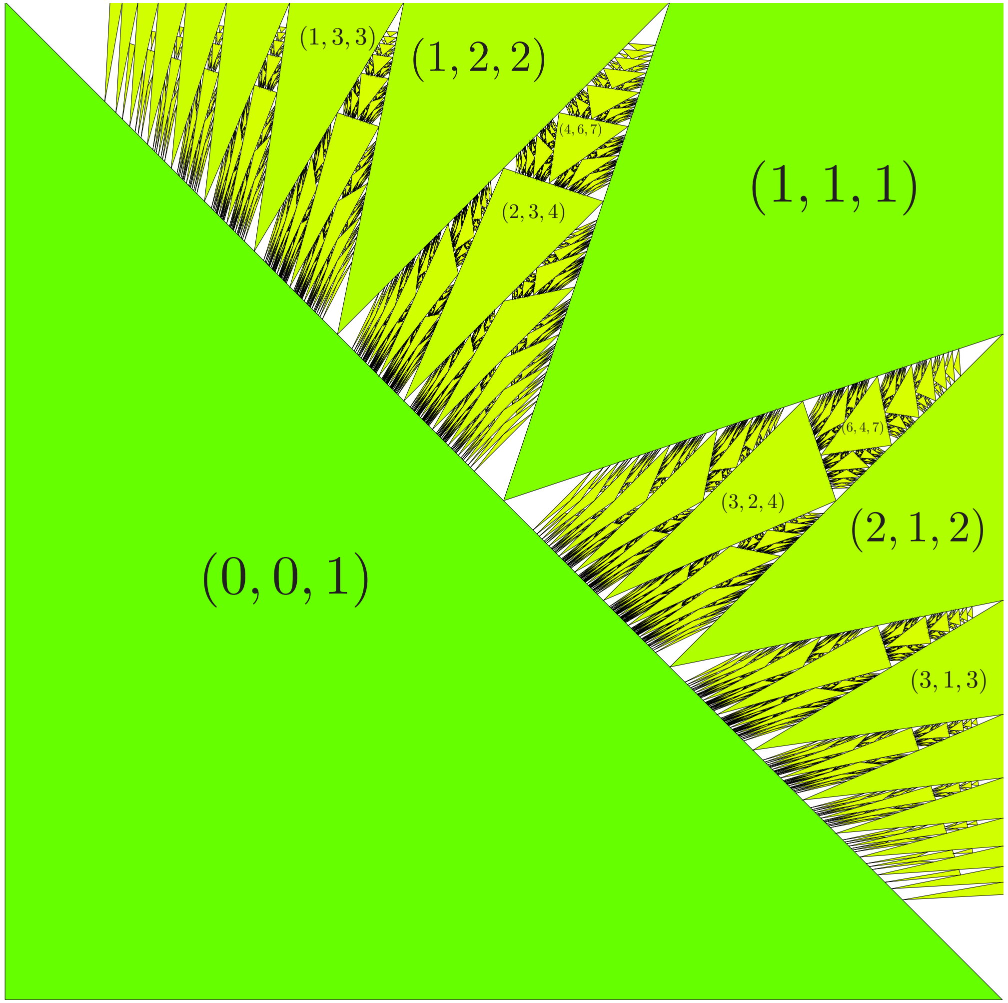


(0, 1, 1)
(0, 2, 3)
(1, 2, 3)
(1, 3, 5)
(0, 1, 2)
(1, 1, 2)
(1, 2, 4)
(1, 1, 3)
(0, 1, 3)
(2, 1, 3)
(1, 1, 4)
(2, 1, 4)
(1, 1, 5)
(3, 1, 5)
(0, 0, 1)
(1, 0, 3)
(1, 0, 2)
(2, 0, 3)
(1, 0, 1)



(0, 1, 1)
(0, 2, 3)
(1, 2, 3)
(1, 3, 5)
(0, 1, 2)
(1, 1, 2)
(1, 2, 4)
(1, 1, 3)
(0, 1, 3)
(2, 1, 3)
(1, 1, 4)
(2, 1, 4)
(1, 1, 5)
(3, 1, 5)
(0, 0, 1)
(1, 0, 3)
(1, 0, 2)
(2, 0, 3)
(1, 0, 1)

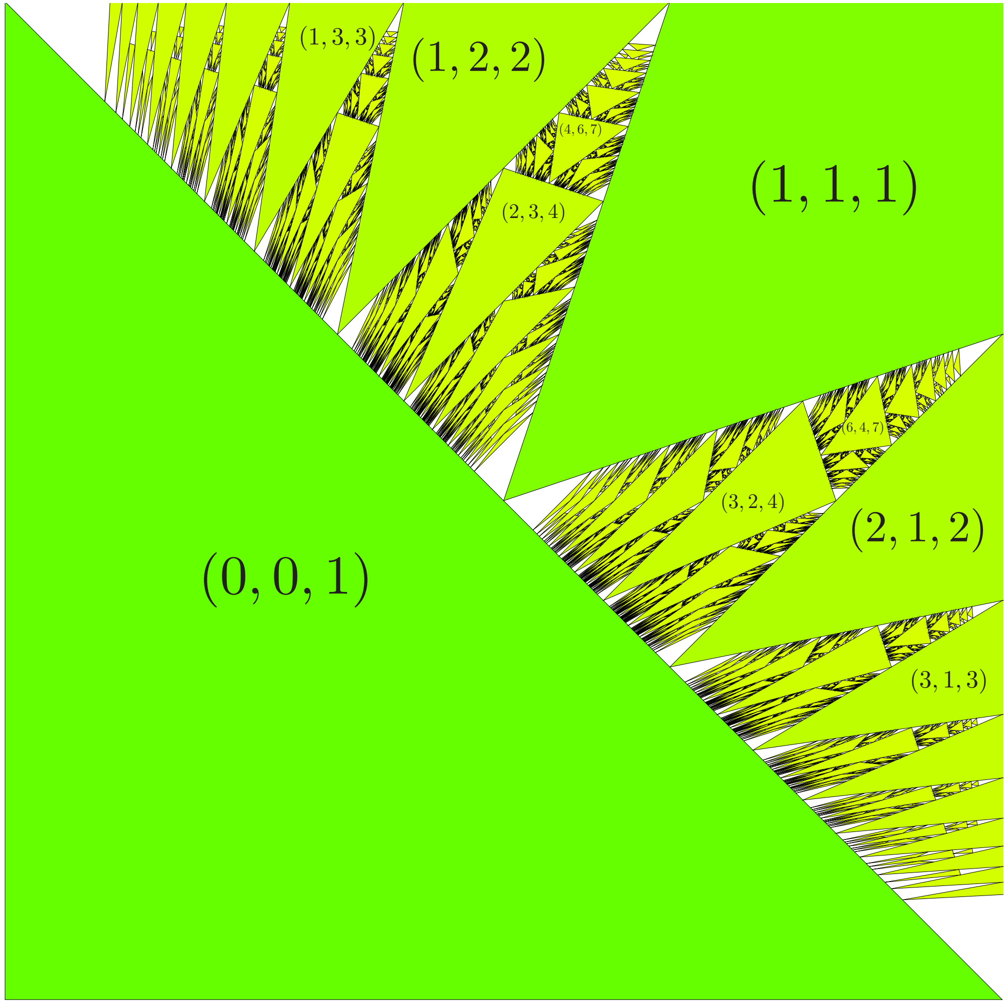
In case of $\{4,6\,|\,4\}$ the initial vertices (in I quad.) are $(1,0,1)$, $(0,1,1)$, $(1,1,0)$



In case of $\{4, 6 \,|\, 4\}$ the initial vertices (in I quad.) are $(1, 0, 1)$, $(0, 1, 1)$, $(1, 1, 0)$

In this picture we show a detail in $[0, 1]^2$ of the 29524 triangles obtained at the 9th iter.

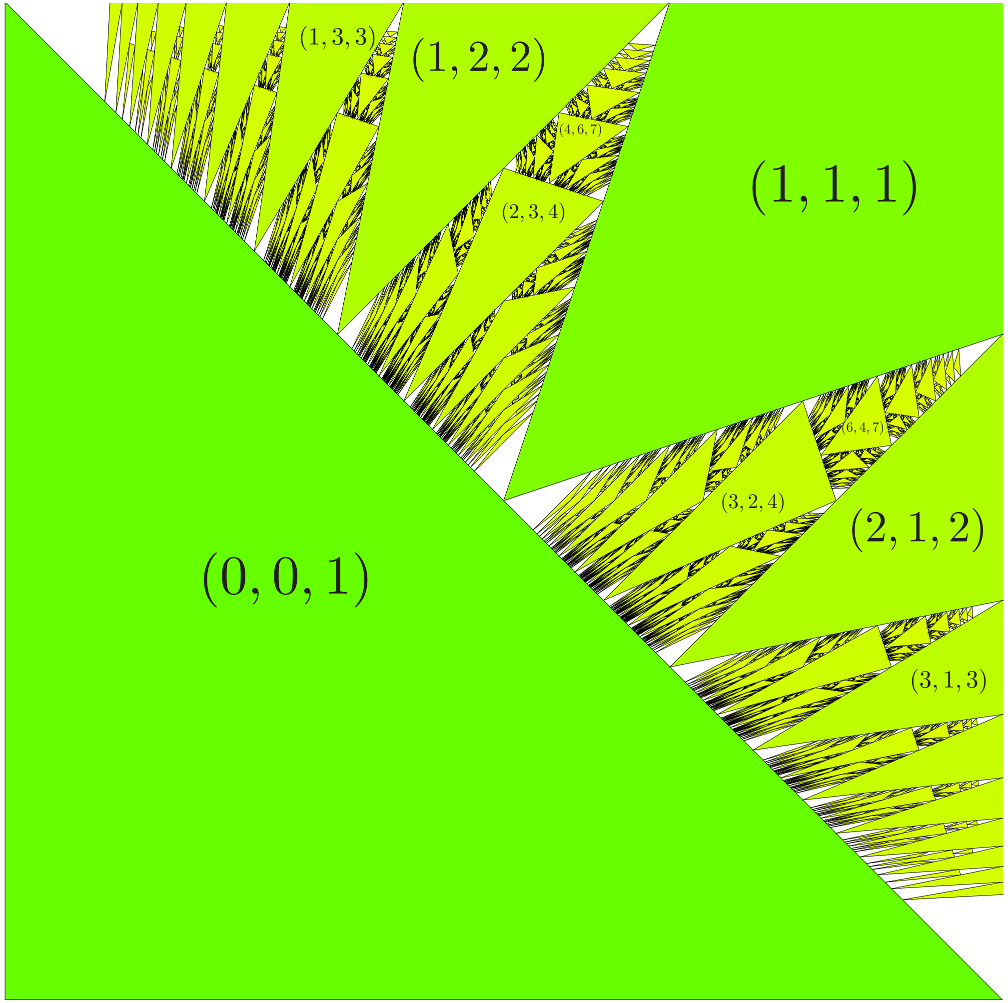
Plane sections of $\{4,6\,|\,4\}$



In case of $\{4,6\,|\,4\}$ the initial vertices (in I quad.) are $(1,0,1)$, $(0,1,1)$, $(1,1,0)$

In this picture we show a detail in $[0,1]^2$ of the 29524 triangles obtained at the 9th iter.

Running time to obtain the 797161 zones obtained after the 12th iter. is ~ 10min.

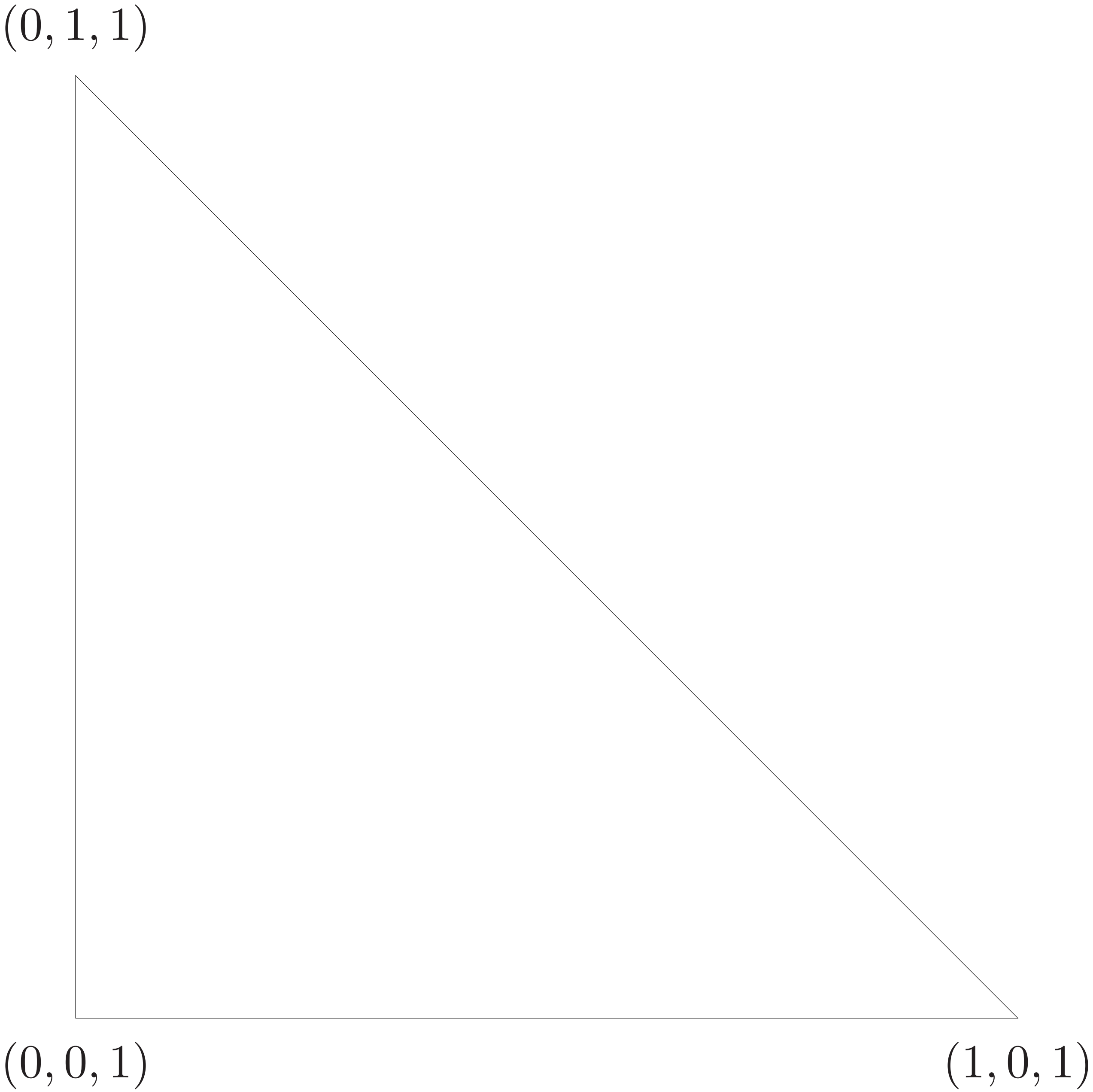


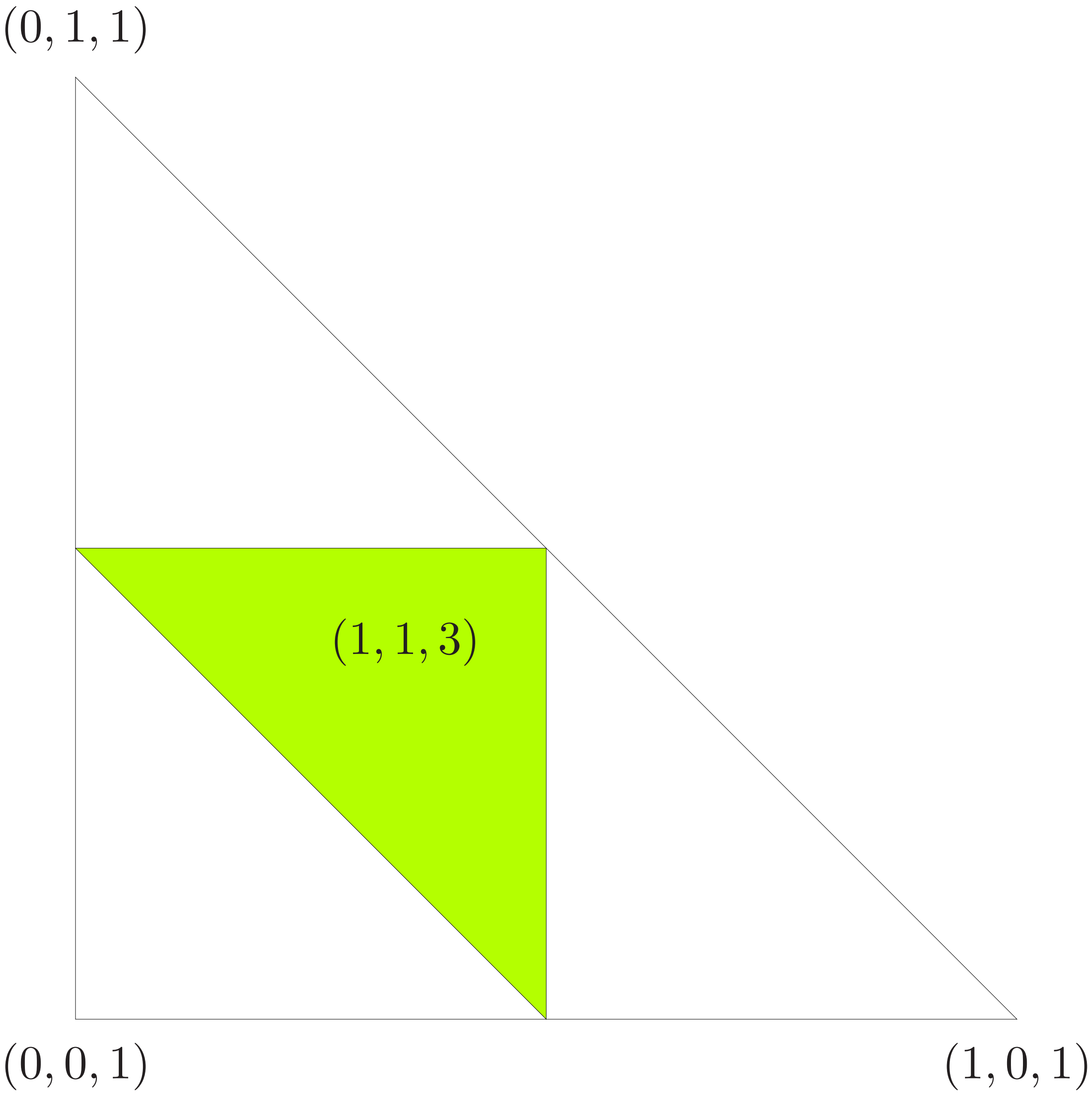
In case of $\{4,6\,|\,4\}$ the initial vertices (in I quad.) are $(1,0,1)$, $(0,1,1)$, $(1,1,0)$

In this picture we show a detail in $[0,1]^2$ of the 29524 triangles obtained at the 9th iter.

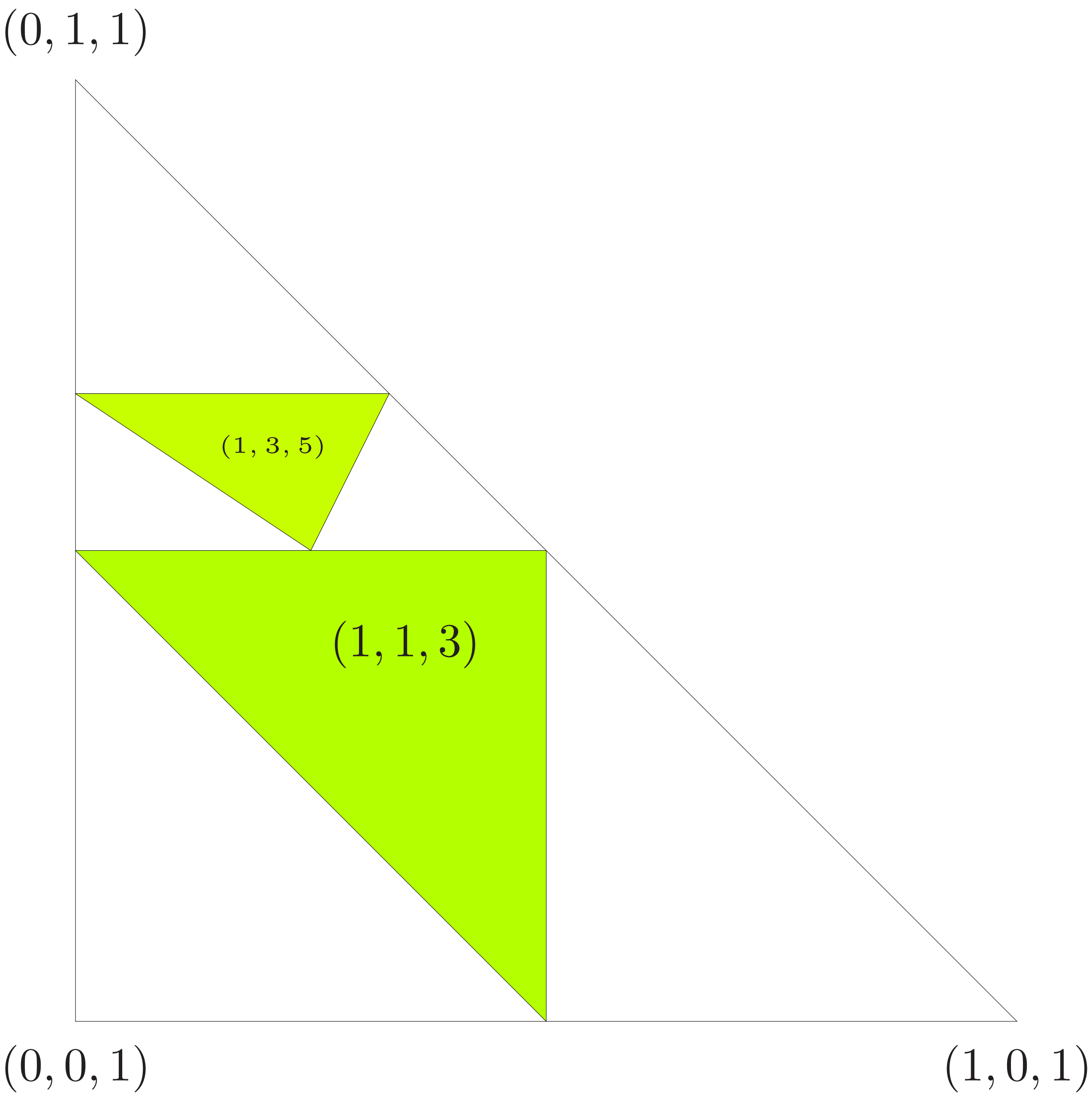
Running time to obtain the 797161 zones obtained after the 12th iter. is ~ 10min.

Note an important property:
the label of every zone, from level 2 on,
is the sum of the labels of the zones.
touching its vertices.

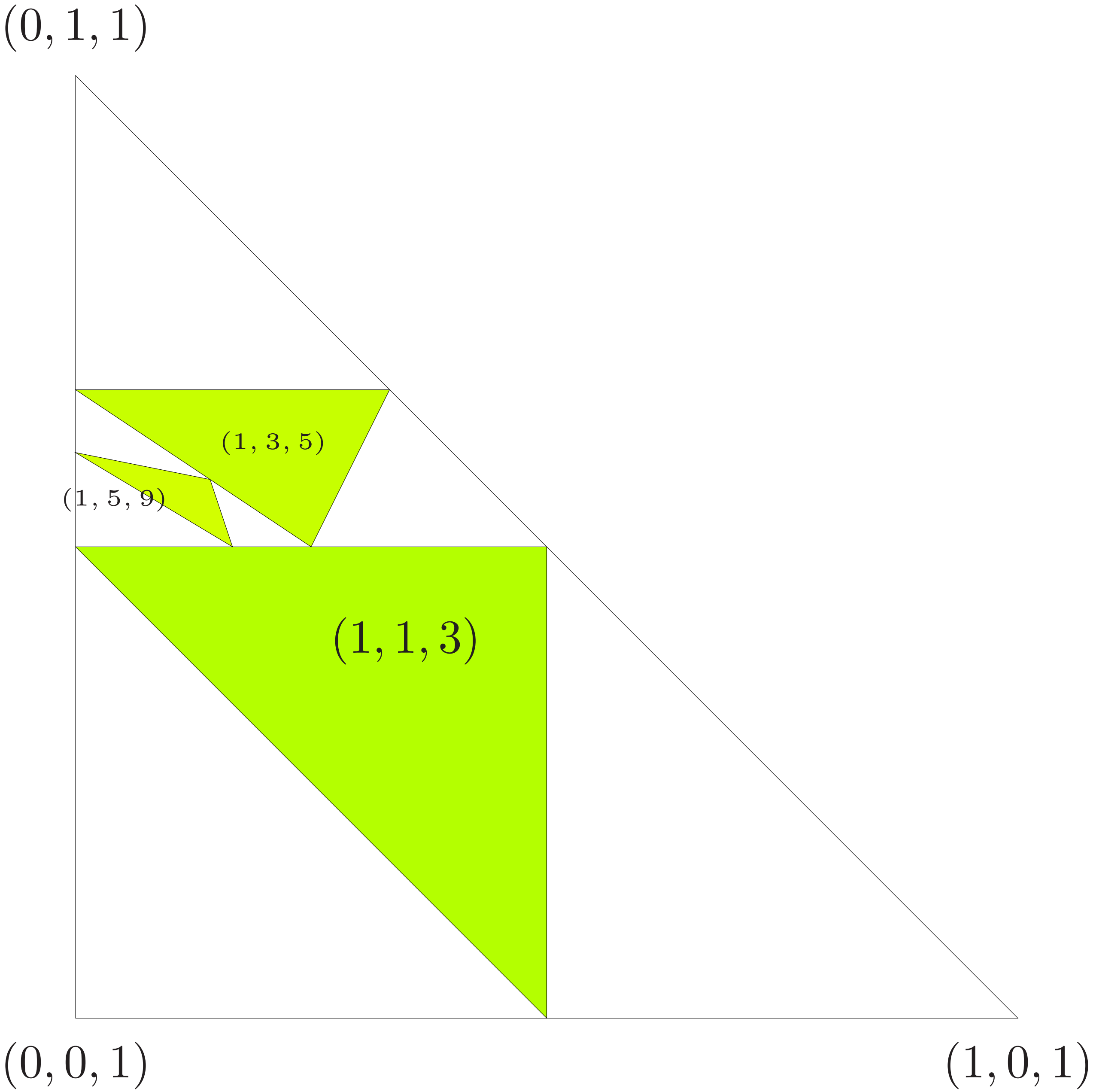

$(0, 1, 1)$
$(0, 0, 1)$
$(1, 0, 1)$



$1 \quad 1 \quad 3$

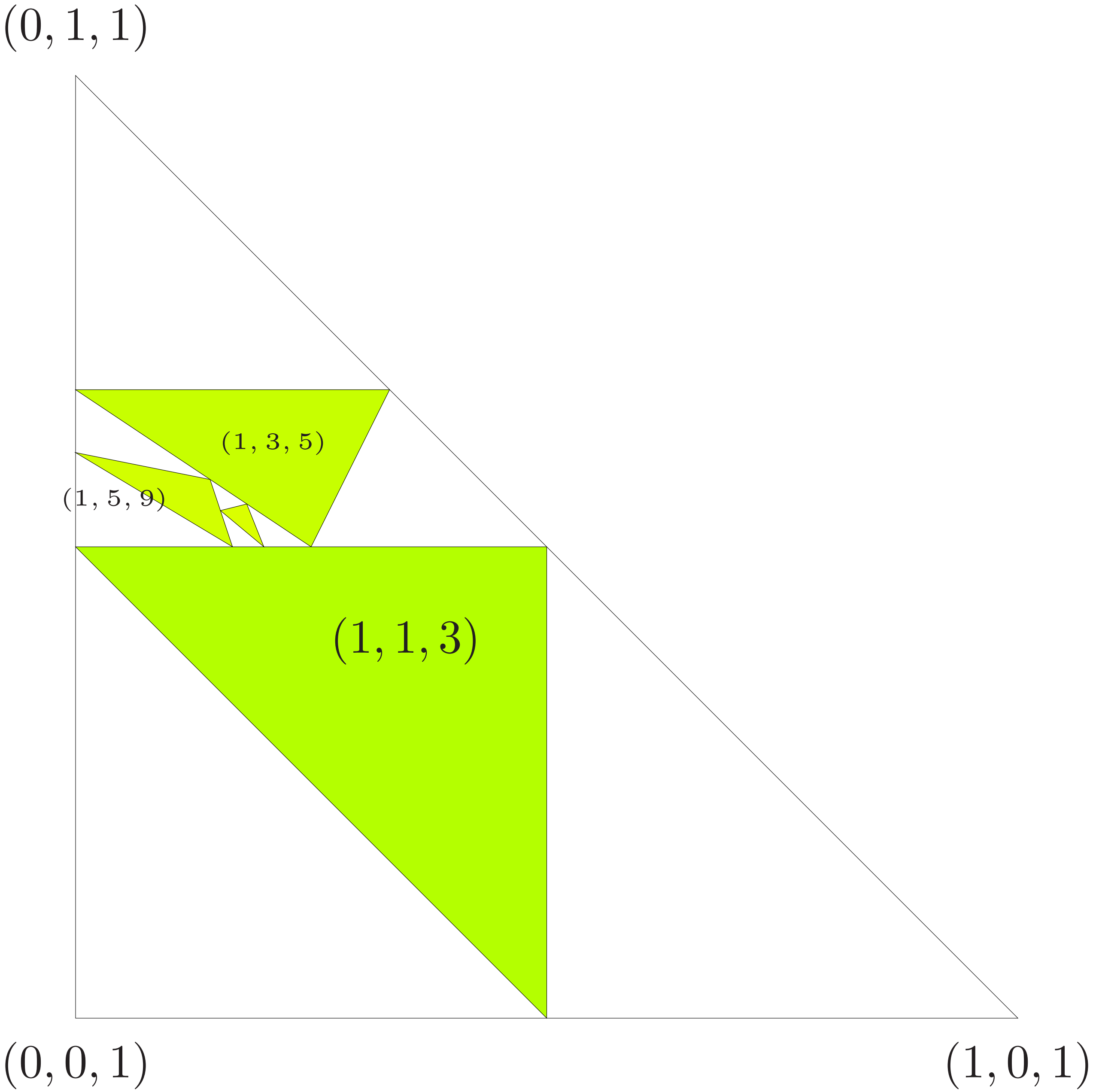


1	1	3
1	3	5

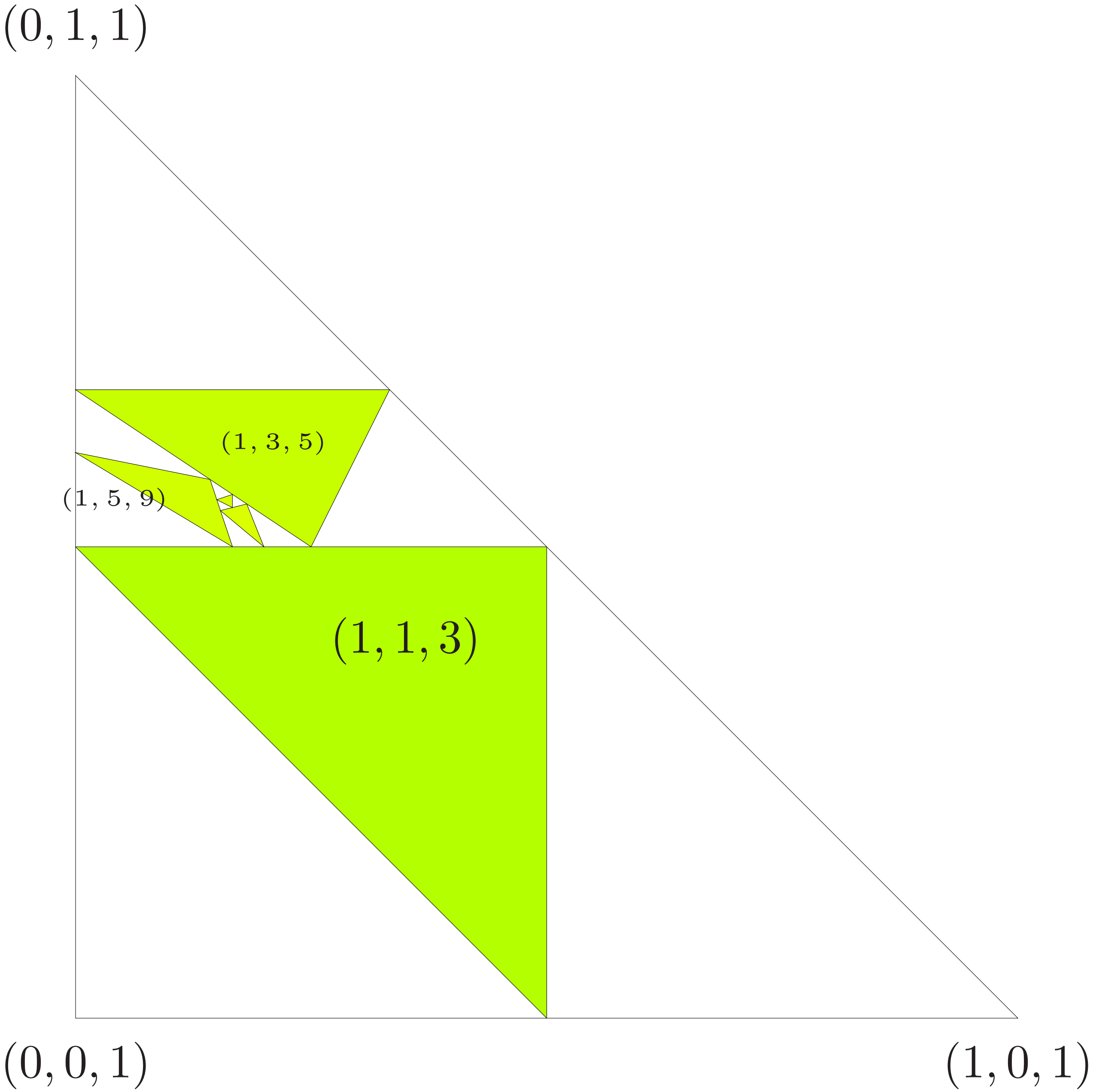


1	1	3
1	3	5
1	5	9

Plane sections of $\{4,6\,|\,4\}$

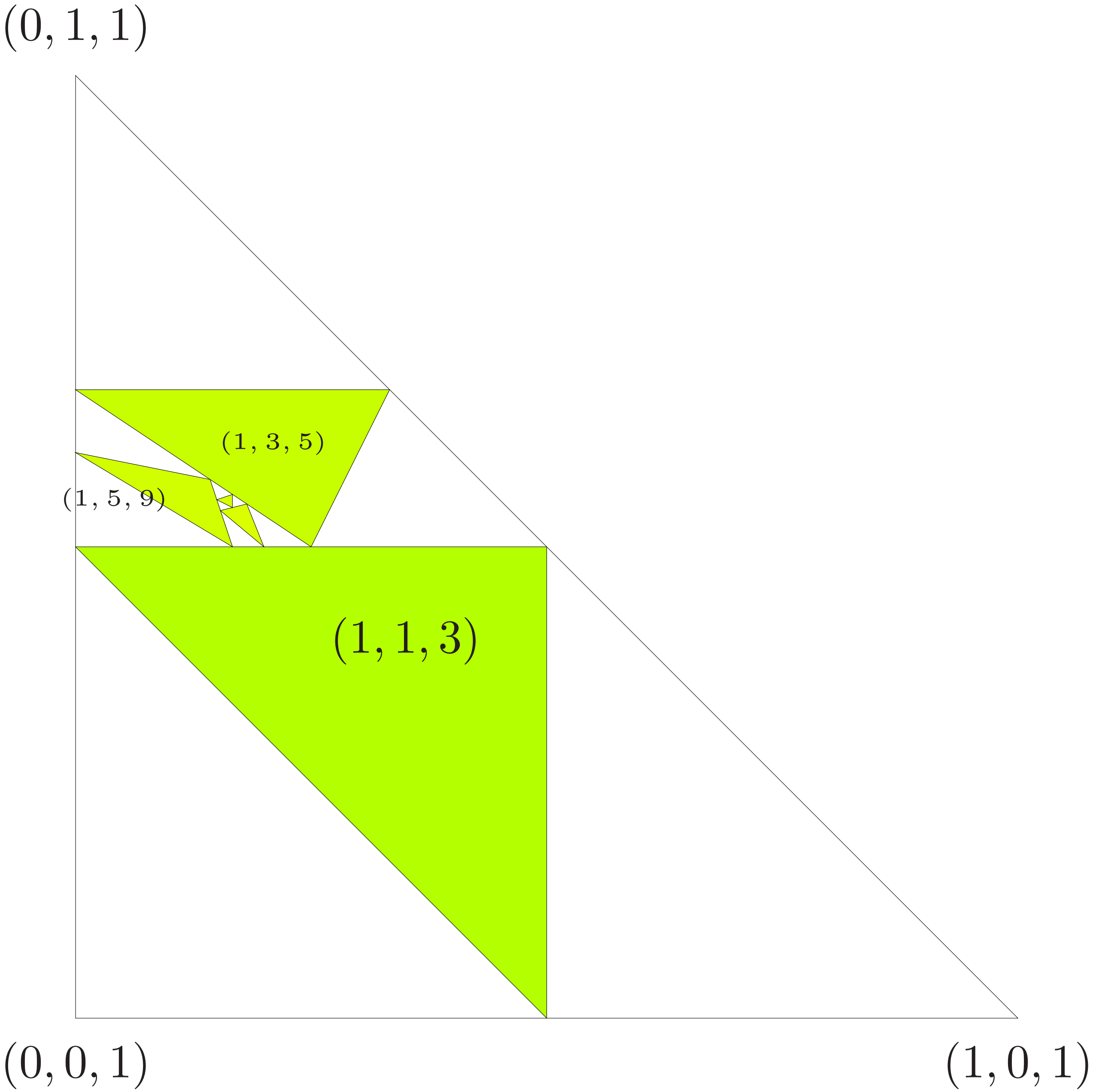


$$\begin{matrix} 1 & 1 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 9 \\ 3 & 9 & 17 \end{matrix}$$



1	1	3
1	3	5
1	5	9
3	9	17
5	17	31

Plane sections of $\{4,6\,|\,4\}$



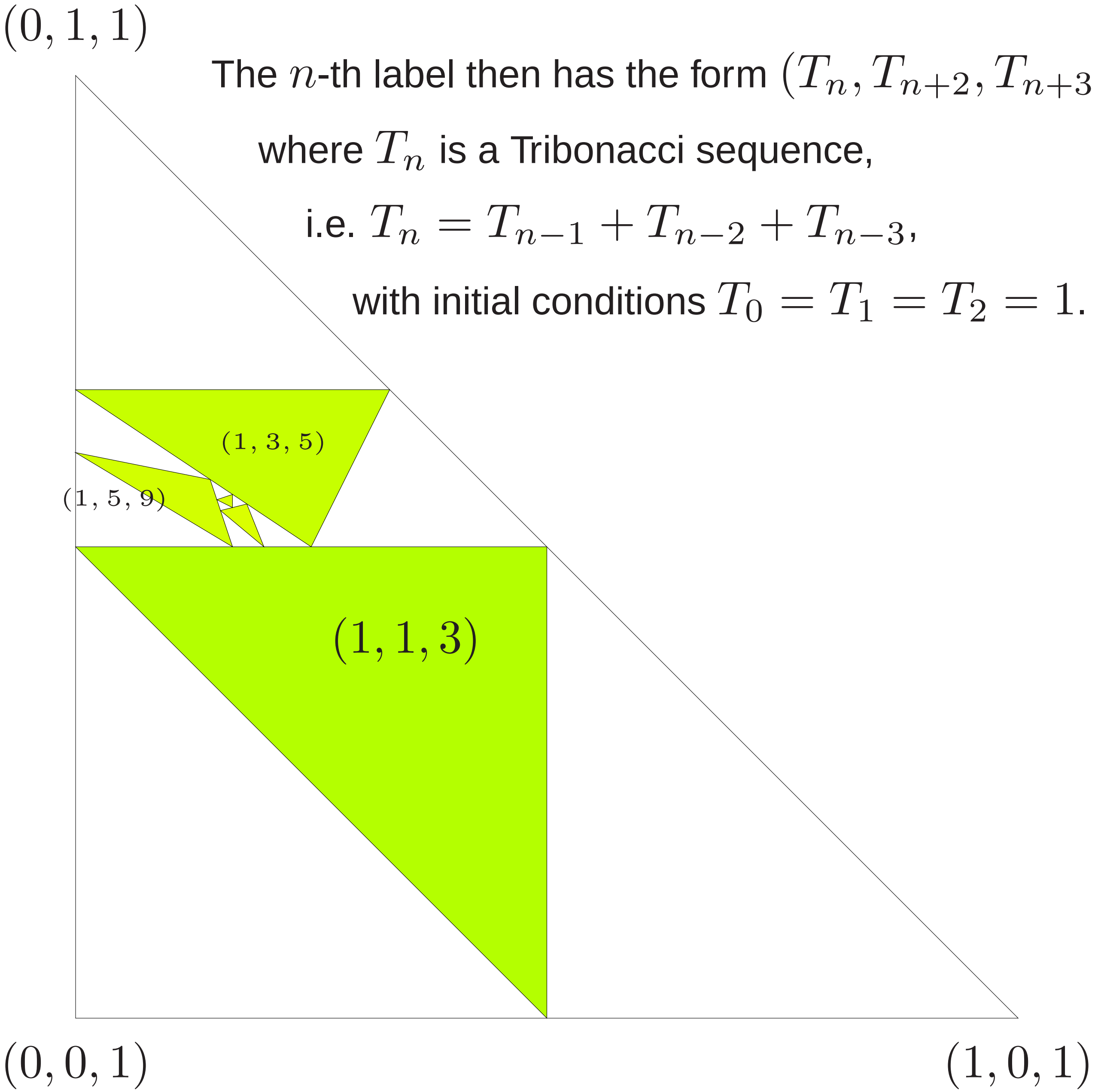
1	1	3
1	3	5
1	5	9
3	9	17
5	17	31
9	31	57
17	57	105

The n-th label then has the form (T_n, T_{n+2}, T_{n+3})

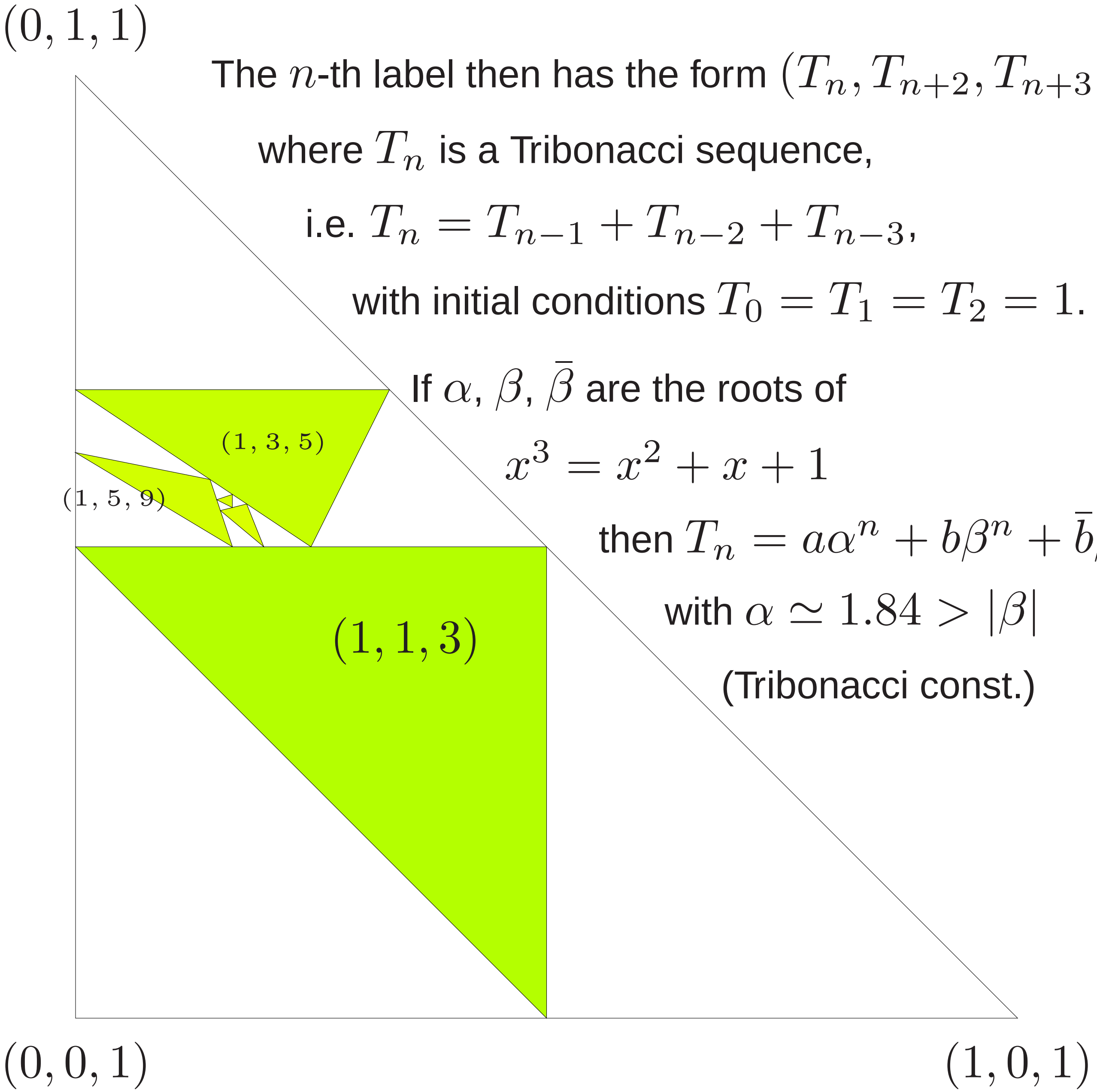
where T_n is a Tribonacci sequence,

i.e. $T_n = T_{n-1} + T_{n-2} + T_{n-3}$,

with initial conditions $T_0 = T_1 = T_2 = 1$.



1	1	3
1	3	5
1	5	9
3	9	17
5	17	31
9	31	57
17	57	105
	⋮	
T_n	T_{n+2}	T_{n+3}



The n-th label then has the form (T_n, T_{n+2}, T_{n+3})

where T_n is a Tribonacci sequence,

i.e. $T_n = T_{n-1} + T_{n-2} + T_{n-3}$,

with initial conditions $T_0 = T_1 = T_2 = 1$.

If α, β, $\bar\beta$ are the roots of

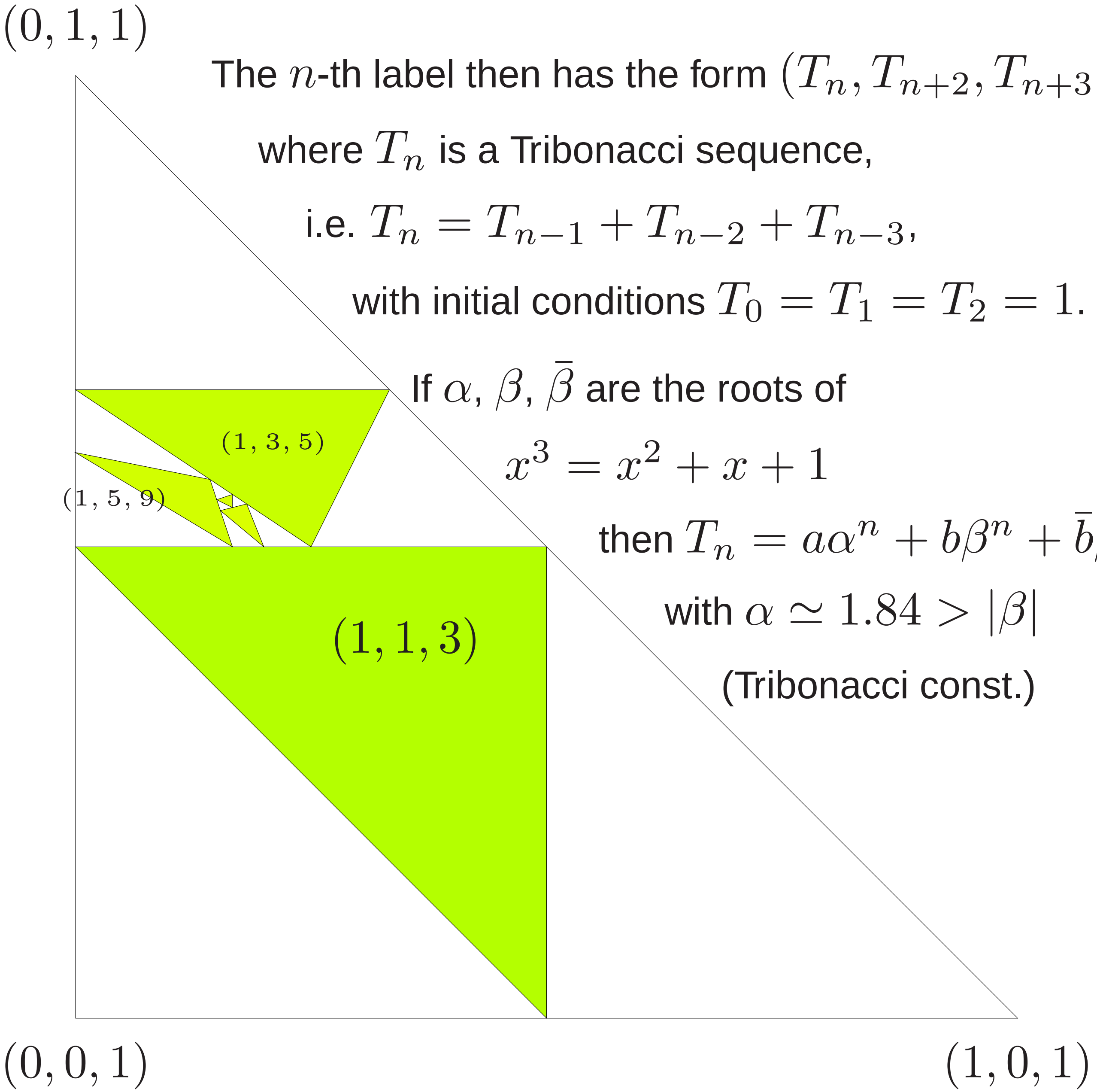
$$x^3 = x^2 + x + 1$$

then $T_n = a\alpha^n + b\beta^n + \bar b\bar\beta^n$

with $\alpha \simeq 1.84 > |\beta|$

(Tribonacci const.)

1	1	3
1	3	5
1	5	9
3	9	17
5	17	31
9	31	57
17	57	105
	⋮	
T_n	T_{n+2}	T_{n+3}



The n-th label then has the form (T_n, T_{n+2}, T_{n+3})

where T_n is a Tribonacci sequence,

i.e. $T_n = T_{n-1} + T_{n-2} + T_{n-3}$,

with initial conditions $T_0 = T_1 = T_2 = 1$.

If α, β, $\bar\beta$ are the roots of

$$x^3 = x^2 + x + 1$$

then $T_n = a\alpha^n + b\beta^n + \bar b\bar\beta^n$

with $\alpha \simeq 1.84 > |\beta|$

(Tribonacci const.)

1	1	3
1	3	5
1	5	9
3	9	17
5	17	31
9	31	57
17	57	105
	$\vdots$	
T_n	T_{n+2}	T_{n+3}
	$\downarrow$	
1	α^2	α^3

fully irrational direction

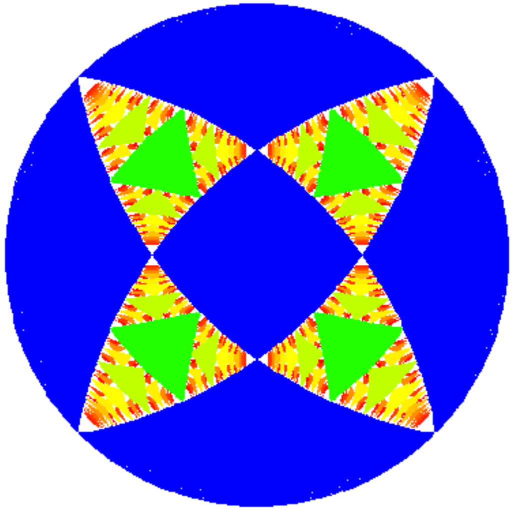
does not belong to any zone or bd

so it must belong to the fractal

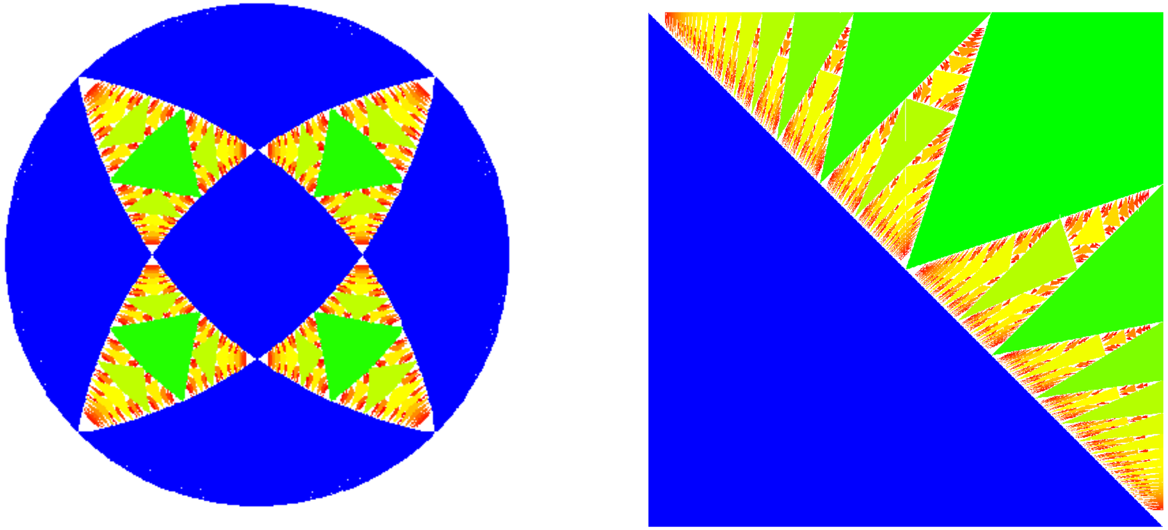
I.A. Dynnikov, R. De Leo

11 – Numerical results

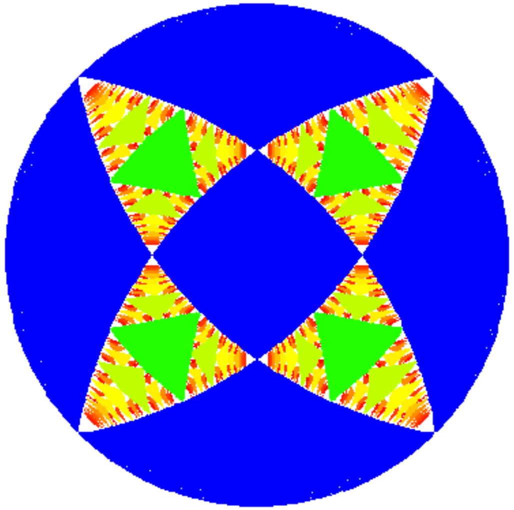
11 – Numerical results

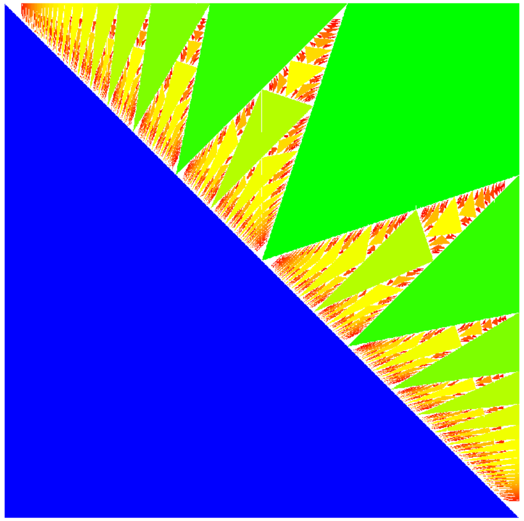


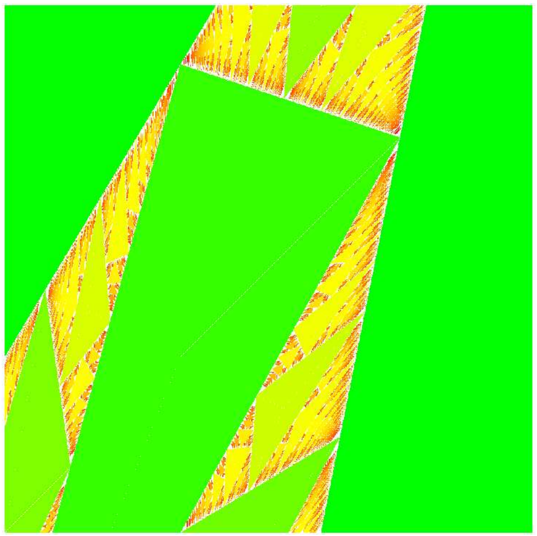
11 – Numerical results



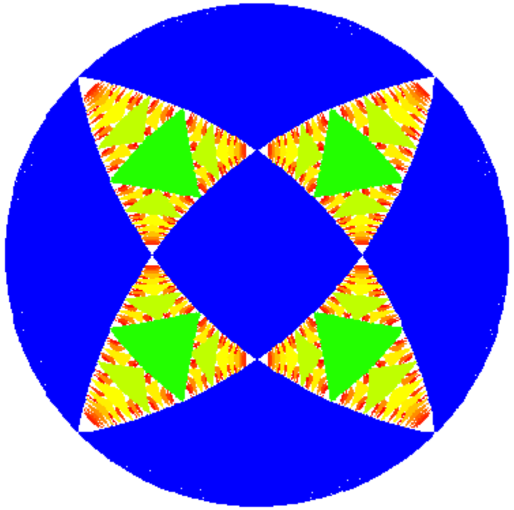
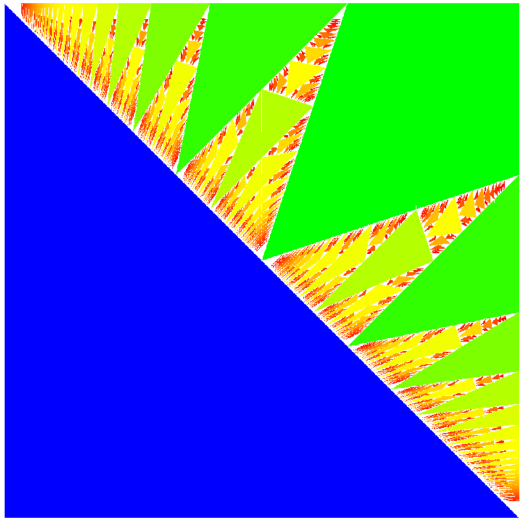
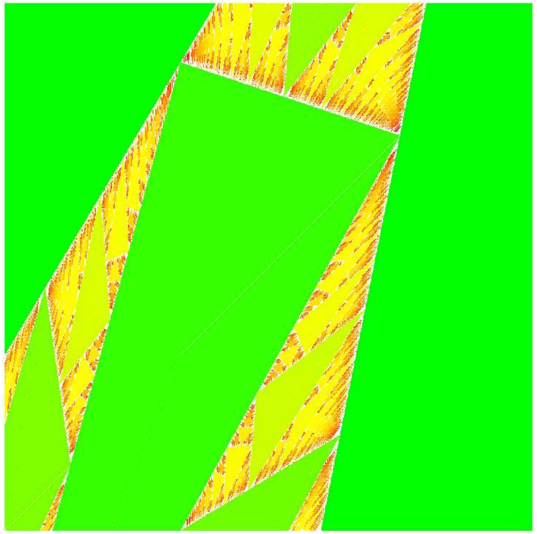
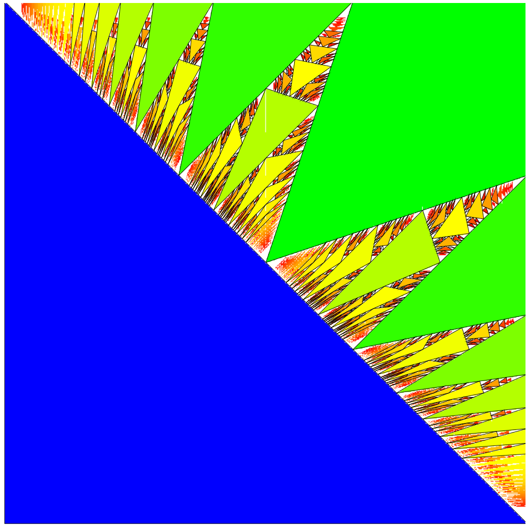
11 – Numerical results



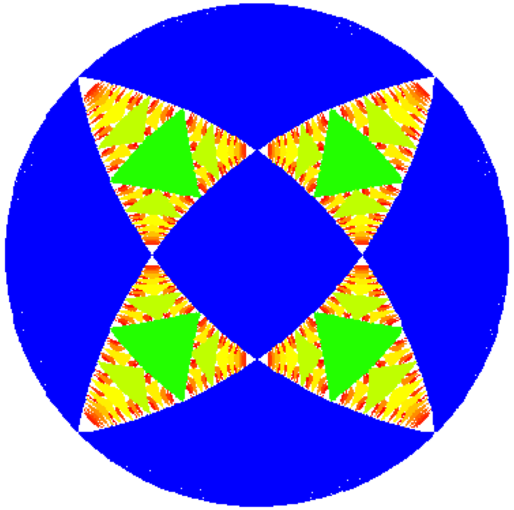


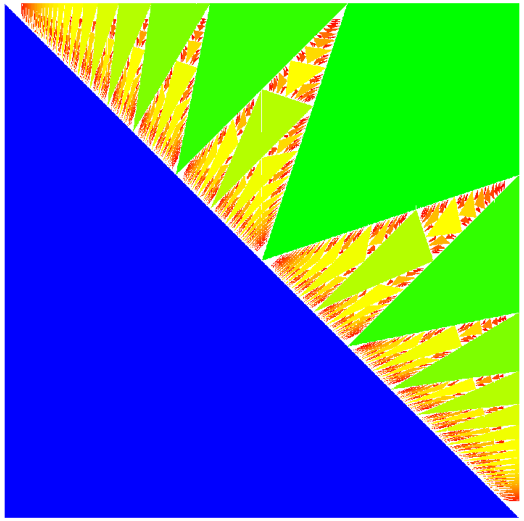


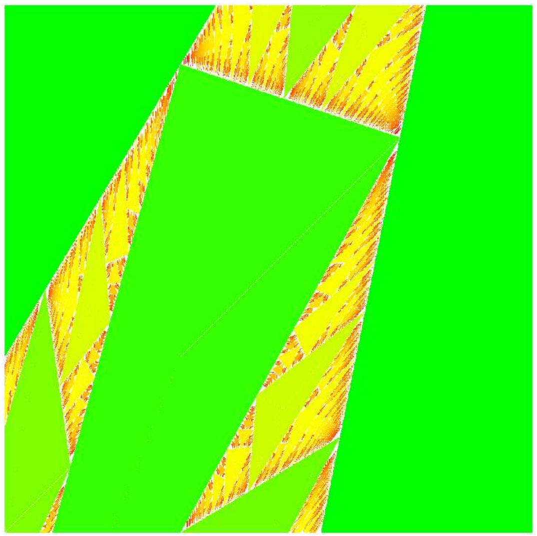
11 – Numerical results

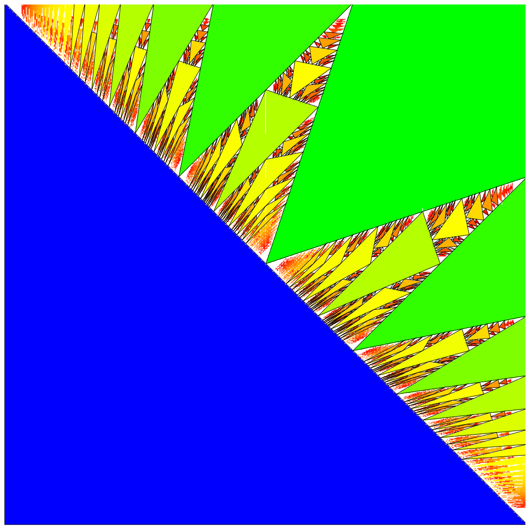





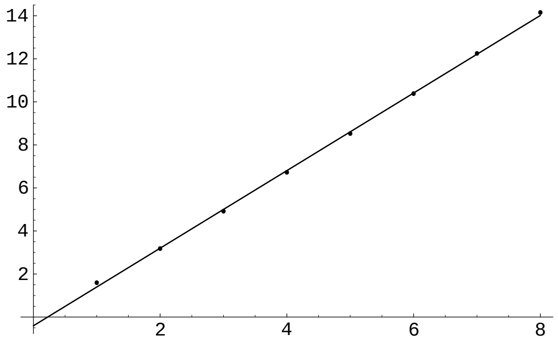
11 – Numerical results











$d_{box} \simeq 1.8$

12 – Conclusions and future developments

- The results for this particular case increased our hopes to prove eventually the conjectures for the stereographic maps of generic hamiltonians.

- It is clear now that the structure of SMs of polyhedra are as rich as the ones of smooth surfaces, so it makes sense to deepen the investigation among this class of surfaces.

- In particular we already started the numerical investigaton of the cubic polyhedron with all “screw vertices” and we will start soon the study of fractals for 3-ply periodic polyhedra of higher genus.

- Moreover, we started the investigation of fractals who are not realized at a single energy level.

13 – Bibliography

- S.P. Novikov, *Hamiltonian formalism and a multivalued analog of Morse theory*, Usp. Mat. Nauk, 37:5 (1982), 3-49
- A.V. Zorich, *A problem of Novikov on the semiclassical motion of electrons in a uniform almost rational magnetic field*, Usp. Mat. Nauk (RMS), 39:5 (1984), 235-236
- I.A. Dynnikov, *The geometry of stability regiones in Novikov's problem on the semiclassical motion of an electron*, RMS, 54:1 (1999), 21-60
- R. De Leo, *Numerical Analysis of the Novikov Problem of a Normal Metal in a Strong Magnetic Field*, SIAM Journal on Applied Dynamical Systems, 2:4 (2003), 517-545, *http://epubs.siam.org/sam-bin/dbq/article/40664*
- R. De Leo, *Topology of plane sections of periodic polyhedra with an application to the Truncated Octahedron*, Experimental Mathematics 15:1 (2006), 109-125 *math.DG/0502219*
- S.P. Novikov, A.Ya.Maltsev *Topology, Quasiperiodic functions and the Transport phenomena*, in "Topology, Quasiperiodic Functions, and the Transport Phenomena" *arXiv:cond-mat/0312710v1*