A quick survey of h-Principle

and isometric embeddings

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Plan of the presentation:

- General results about isometric immersions

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- The Nash-Gromov Implicit Function Theorem

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- General results about isometric immersions
- The Nash-Gromov Implicit Function Theorem
- Inducing structures on manifolds

Main Sources

• D. Spring, “The golden age of immersion theory in topology: 1959-1973”, http://arxiv.org/abs/math/0307127 (2003)

• M. Gromov, V. Rokhlin, “Embeddings and immersions in Riemannian geometry”, Russian Mathematical Surveys, 25:5 (1970)

• M. Gromov, “Partial Differential Relations”, Springer, 1983

• M. Gromov, “Geometric, Algebraic and Analytic Descendants of Nash Isometric Embedding Theorems”, 2015, http://www.ihes.fr/~gromov/PDF/nash-copy-Oct9.pdf

• R. De Leo, ”A note on non-free isometric immersions”, Russian Math Surveys, 63:3 (2010)

• G. D’Ambra, R. De Leo, A. Loi, “Partially isometric immersions and free maps”, Geometriae Dedicata, 151:1 (2011)

• R. De Leo, “On some geometrical and analytical problems arising from the theory of Isometric Immersion”, 2011, http://veprints.unica.it/584/1/PhD_RobertoDe_Leo.pdf

Isometric Embeddings

The isometric immersions problem

Question. Given a C^k Riemannian metric $g = g_{\alpha\beta}(x)\, dx^\alpha \otimes dx^\beta$ on M^n, for which q and k' can we find a (global or local) $C^{k'}$ isometric immersion $f : M \to \mathbb{R}^q$?

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Namely, can we find q (global or local) functions $f^a \in C^{k'}(M)$ such that

$$\delta_{ab}\, \partial_\alpha f^a(x)\, \partial_\beta f^b(x) = g_{\alpha\beta}(x)?$$

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Remark: for $n = 2$ and $q = 3$, we can riarrange the coordinates so that $f(x,y,z) = (x,y,z(x,y))$ and the equation above is equivalent to the following Monge-Ampére type eq., known as Darboux equation:

$$\det\left(\nabla_\alpha(\nabla_\beta z)\right) = K(\det g)(1 - \|\mathrm{grad}(z)\|^2)$$

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It turns out that the answer depends on the regularity.

Local results: Analytical case

C^{an}-Local Conjecture (Schlæfli (1873)[1])

Every 2-dimensional analytical Riemannian manifold admits analytical local isometric embeddings into $\mathbb{R}^3$*.*

[1] L. Schaefli, “Nota alla memoria del Sig. Beltrami sugli spazi di curvatura costante”, Ann. di Mat., 5 (1873), 170-193

[3] M. Janet, “Sur la possibilité de plonger un espace Riemannien donné dans une espace Euclidiéen”, Annal. Soc. Polon. Math., 5 (1926), 38-43

[3] E. Cartan, “Sur la possibilité de plonger un espace riemannien donné dans un espace euclidéen”, Ann. Soc. Polon. Math. , 6 (1927), 17

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C^{an}-Local Theorem (Janet (1926)[2], Cartan (1927)[3])

Every n-dimensional analytical Riemannian manifold admits analytical local isometric embedding into $\mathbb{R}^{s_n}$, $s_n = n(n+1)/2$.

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Local results: Smooth case

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This problem is still open, e.g. see:

- Yau,"Problem Section, Seminar on Diff. Geom.", Ann. of Math. Studies 102, Princeton University Press (1982)
- Lin,"The local isometric embedding in $\mathbb{R}^3$ of 2-dim. Riem. mfds with non-neg. curv", J. of Diff. Geom. 21 (1985), 213-230
- Hong, Zuily, "Existence of C^∞ local solutions for the Monge-Ampére equation", Inv. Math. 89 (1987), 645-661
- Han, Hong, "Isometric Embedding of Riemannian Manifolds in Euclidean Spaces", 2006, Math. Surv. and Monographs, AMS
- Han, "Isometric Embeddings of Surfaces in $\mathbb{R}^3$", Recent Developments in Geometry and Analysis (2012), 113-145

Local results: General case

The following are corollary of global results of Nash and Kuiper that we are going to present shortly:

Theorem (Nash (1954), Kuiper (1955))

Every C^1 Riemannian n-manifold admits C^1 local isometric immersions into $\mathbb{R}^{n+1}$.

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Remark 1: the case $r = 2$ is still open.

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Remark 1: the case $r=2$ is still open.

Remark 2: Gromov improved the second result to $q=n^2+10n+3$ for $r=3$ and $q=s_{n+2}$ for $r\geq 4$.

Local results: General case

Conjecture (Gromov (2015))

Every C^r parallelizable Riemannian n-manifold admits C^r local isometric immersions into $\mathbb{R}^q$ for $q = s_n + 1$ and $r = 1, 2, \cdots, \infty, an$.

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Conjecture (Gromov (2015))

Let f be a global analytical section of the bundle $\mathcal{F}(M)$ of frames over the parallelizable n-manifold M. Then there exists $f \in C^{an}(M, \mathbb{R}^{s_n+1})$ such that $f_* f$ is an orthonormal $(s_n + 1)$-frame in $\mathbb{R}^{s_n+1}$.

Global results

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Let g be a C^r Riemannian metric on M^n, $r = 3, 4, \ldots, \infty$. Then there exist C^r isometries of (M, g) into $\mathbb{R}^q$ for $q = 3s_n + 4n$ if M is compact and into $q = (n+1)(3s_n + 4n)$ if M is open.

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Let g be a C^{an} Riemannian metric on M^n. Then there exist C^{an} isometries $f : (M, g) \to \mathbb{R}^q$ for $q = 3s_n + 4n$.

Open Problems and Conjectures

Question

Do there exist C^{an} or C^∞ Riemannian n-manifolds admitting C^r isometric immersions into $\mathbb{R}^q$ for some q_r but no C^{an} or C^∞ isometric immersions for $q \leq (1 + c_r)q_r$ for some $c_r > 0$?

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Conjecture

If $q > 0.36n^2 + 1.36n$, all C^2 Riemannian n-manifolds admit global isometric C^2 immersions into $\mathbb{R}^q$

The Nash-Gromov Implicit Function Theorem

Partial Differential Operators

Let $F \xrightarrow{\pi_F} E$ a C^∞-fibration and $G \xrightarrow{\pi_G} E$ a vector bundle.
Let $\Gamma^r F$ the C^r sections of $F \xrightarrow{\pi_F} E$ and similarly for $\Gamma^0 G$.

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Definition

A C^k PDO over F of order r with values in G is a map

$$\mathcal{L}_r : \Gamma^r F \to \Gamma^0 G$$

whose coeffs, in any coord system, are all C^k and whose value on a section $f \in \Gamma^r F$ at a point $x \in E$ depends only on $j^r_x f$.

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In coordinates (x^α, f^i) and (x^α, g^a), $\mathcal{L}_r$ writes as

$$\mathcal{L}_r(f)(x^\alpha) = (\Lambda^a_r(x^\alpha, f^i(x^\alpha), \partial_\alpha f^i(x^\alpha), \dots, \partial_{\alpha_1 \dots \alpha_r} f^i(x^\alpha)) .$$

where $\Lambda_r = (\Lambda^a_r) : J^r F \to G$ is some C^k map.

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The equation

$$\mathcal{L}_r(f) = \phi$$

is then equivalent to

$$\Lambda_r^a(x^\alpha, f^i(x^\alpha), \partial f^i_\alpha(x^\alpha), \dots, \partial f^i_{\alpha_1 \dots \alpha_r}(x^\alpha)) = \phi^a(x^\alpha)$$

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The corresponding map $\Lambda_\xi : J^1(M, \mathbb{R}) \to M \times \mathbb{R}$ is defined as

$$\Lambda_\xi(x^\alpha, f, f_\alpha) = \xi^\alpha(x) f_\alpha \,.$$

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The corresponding PDE

$$(j^1 f)^* \Lambda_\xi = \phi$$

is called *cohomological equation*. In coordinates writes as

$$\xi^\alpha(x) \partial_\alpha f(x) = \phi(x)$$

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$$\mathcal{D}_{M,q} : C^1(M, \mathbb{R}^q) \simeq \Gamma^1 F \to \Gamma^0(S^0_2 M)$$

defined as $\mathcal{D}_{M,q}(f) = f^* e_q$ is also a PDO of order 1.

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In coordinates

$$\mathcal{D}_{M,q}(f) = \delta_{ij} \partial_\alpha f^i \partial_\beta f^j\,,$$

so that the corresponding map $\Lambda_{M,q} : J^1(M, \mathbb{R}^q) \to S^0_2 M$ is defined as

$$\Lambda_{M,q}(x^\alpha, f, f_\alpha) = \delta_{ij} f^i_\alpha f^j_\beta\,.$$

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$$T_{x_0}\pi_F(\eta_f(x_0)) = T_{x_0}\pi_F\left(\left.\frac{df_t(x_0)}{dt}\right|_{t=0}\right) =$$

$$= \left.\frac{d(\pi_F \circ f_t)(x_0)}{dt}\right|_{t=0} = \left.\frac{dx_0}{dt}\right|_{t=0} = 0\,,$$

namely $\eta_f \in \Gamma^r_f$.

Linearization of a PDO

Definition

The *linearization of* $\mathcal{L}_r$ *at* f is the *linear* PDO

$$\ell_{r,f} : \Gamma^r_f \to \Gamma^0 G$$

defined by

$$\ell_{r,f}(\eta) = \frac{d}{dt}\mathcal{L}_r(f_t)\Big|_{t=0}$$

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The PDO

$$\ell_r : \Gamma^r(VF) \to \Gamma^0 G,$$

defined as $\ell_r(f,\eta) = \ell_{r,f}(\eta)$, is the *tangent map* (or *differential*) of $\mathcal{L}_r$.

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The Lie derivative L_ξ is linear and so it is to be expected that its differential ℓ_ξ is identical to it. Indeed

$$\ell_\xi(f, \delta f) = \delta L_\xi(f) = \delta(\xi^\alpha \partial_\alpha f) = \xi^\alpha \partial_\alpha \delta f$$

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The isometric operator $\mathcal{D}_{M,q}$ instead is quadratic and its differential $\ell_{M,q}$ is

$$\ell_{M,q}(f,\delta f) = \delta \mathcal{D}_{M,q}(f) = \delta(\delta_{ij}\partial_\alpha f^i \partial_\beta f^j) = 2\delta_{ij}\partial_\alpha f^i \partial_\beta \delta f^j$$

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We say that $\mathcal{L}_r$ is *infinitesimally invertible* of defect $d \geq r$ and order s over some subset $\mathcal{A} \subset \Gamma^r F$ if there exist a family of *linear* PDOs $m_f : \Gamma^s G \to \Gamma^0_f(VF)$ of some order s, with $f \in \mathcal{A}$, satisfying the following properties:

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3. $\ell_r(m(f, \rho)) = \rho$ for every $f \in \Gamma^{r+d} F$ and $\rho \in \Gamma^{r+s} G$.

Linearization of a PDO

Example

The isometric operator $\mathcal{D}_{M,q} : C^1(M, \mathbb{R}^q) \to S_n^0(M)$ admits an infinitesimal inverse of defect 2 and order 0 over the space of free maps $F^2(M, \mathbb{R}^q)$.
Indeed we know that the linearized equation $\ell_{M,q}(f) = \delta g$ can be solved *algebraically* over the set of free maps. Let $\delta f_{f,\delta g}$ be the solution closest to the origin in some metric and set $m(f, \delta g) = \delta f_{f,\delta g}$. Clearly $\ell_{M,q}(m(f, \delta g)) = \delta g$.

The Implicit Function Theorem

Theorem (Nash, Gromov)

Let $\mathcal{L}_r$ be a C^k PDO of order r admitting an infinitesimal inverse of order s and defect d over some subset $\mathcal{A} \subset \Gamma^r F$ and set $\hat{s} = \max(d, 2r+s)+s+1$.
Then, for every $f_0 \in \mathcal{A} \cap \Gamma^\infty F$, there is a neighbourhood $\mathcal{U} \subset \Gamma^{\hat{s}} G$ of 0 such that, for every $\rho \in \mathcal{U} \cap \Gamma^{s'} G$ with $s' \geq \hat{s}$, the equation $\mathcal{L}_r(f) = \mathcal{L}_r(f_0) + \rho$ has a $C^{s'-s}$ solution.

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Corollary

Let $\mathcal{L}_r$ a PDO infinitesimally invertible over $\mathcal{A} \subset \Gamma^r F$. Then the restriction of $\mathcal{L}_r$ to $\mathcal{A} \cap \Gamma^\infty F$ is an open map.

Application: Nash Theorem

Theorem (Nash)

If $g_0 = \mathcal{D}_{M,q}(f_0)$ with $f_0 \in Free^\infty(M, \mathbb{R}^q)$, then the C^s metric $g_0 + g$, $s \geq 3$, can be realized by a C^s immersion f for every C^3-small enough g.

Indeed in this particular case $r = 1$, $s = 0$ and $d = 2$, so that $\hat{s} = \max(d, 2r + s) + s + 1 = 3$.

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Hence the IFT theorem implies that,
for every $f_0 \in Free^2(M,\mathbb{R}^q) \cap C^\infty(M,\mathbb{R}^q) = Free^\infty(M,\mathbb{R}^q)$, there is a neighbourhood $\mathcal{U} \subset \Gamma^3(S^0_2 M)$ of 0 such that, for all $g \in \mathcal{U}$ of class $C^{s'}$, $s' \geq 3$, the equation $f^* e_q = g_0 + g$ has a solution of class $C^{s'}$.

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Partial Differential Relations

Let $F \xrightarrow{\pi_F} E$ a C^∞-fibration.

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A *Partial Differential Relation* of order r is a subset $\mathcal{R} \subset J^r F$

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Sections of $\varphi : E \to J^r F$ such that $\varphi = j^r f$ are called holonomic

Obstructions to the existence of solutions of a PDR (in particular of a PDE) can be of *topological* or *analytical* origin. In the first case not even *formal* solutions exist. When formal solutions exist, analytical ones may or may not exist.

Partial Differential Relations

Example

Consider the case $E = \mathbb{R}^2$ (x, y), $F = \mathbb{R}^2 \times \mathbb{R}$ (x, y, z) and $J^1(\mathbb{R}^2, \mathbb{R})$ (x, y, z, p_x, p_y).

Given a vector field $\xi = (\xi_x(x, y), \xi_y(x, y))$ on $\mathbb{R}^2$, the 1st order PDE $L_\xi f = g$ is represented by the hypersurface

$$\mathcal{R} = \{\xi_x(x, y)p_x + \xi_y(x, y)p_y = g(x, y)\} \subset J^1(\mathbb{R}^2, \mathbb{R})$$

If ξ is never zero, then there is no obstruction for its formal solvability for any $g \in C^0(\mathbb{R}^2)$, e.g. take

$$\varphi(x, y) = \left(x, y, z(x, y), g(x, y)\frac{\xi_x(x,y)}{\|\xi(x,y)\|}, g(x, y)\frac{\xi_y(x,y)}{\|\xi(x,y)\|}\right)$$

Depending on the topology of the integral trajectories of ξ, though, true solutions might not exist.

E.g. $\mathcal{R} = \{2yp_x + (1 - y^2)p_y = 1\}$ admits no holonomical C^1 section, namely there is no C^1 function f st

$$2y\partial_x f(x, y) + (1 - y^2)\partial_y f(x, y) = 1.$$

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Remark

Since the fibers of $\mathcal{R}^{k+1} \to \mathcal{R}$ are all contractible, if $\mathcal{R}$ is stable then every section $\varphi : E \to \mathcal{R}$ lifts to a section $\varphi : E \to \mathcal{R}^k$ unique modulo homotopies.

Prolongations of the isometry PDR

Example

Consider the case $E = M^n$ (x^α), $F = M^n \times \mathbb{R}^q$ (x^α, f^a) and $J^1(M^n, \mathbb{R}^q)$ $(x^\alpha, f^a, v^a_\alpha)$.

The isometric PDR $\mathcal{I}_g$ is the closed subset of $J^1(M^n, \mathbb{R}^q)$ of codimension s_n defined by the system

$\delta_{ab} v^a_\alpha v^b_\beta = g_{\alpha\beta}(x)$.

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Its 1-prolongation $\mathcal{I}^1_g$ is the closed subset of $J^2(M^n, \mathbb{R}^q)$ $(x^\alpha, f^a, v^a_\alpha, v^a_{\alpha\beta})$ of codimension $(n+1)s_n$ defined by the system

$\delta_{ab} v^a_\alpha v^b_\beta = g_{\alpha\beta}(x)$

$\delta_{ab}(v^a_{\alpha\mu} v^b_\beta + v^a_\alpha v^b_{\beta\mu}) = \partial_\mu g_{\alpha\beta}(x)$

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$\delta_{ab}(v^a_{\alpha\mu} v^b_\beta + v^a_\alpha v^b_{\beta\mu}) = \partial_\mu g_{\alpha\beta}(x)$
equivalent to
$\delta_{ab} v^a_\alpha v^b_\beta = g_{\alpha\beta}(x)$,
$\delta_{ab} v^a_\alpha v^b_{\beta\gamma} = g_{\alpha\lambda}(x)\Gamma^\lambda_{\beta\gamma}(x)$, $\Gamma^\lambda_{\beta\gamma} = \frac{1}{2} g^{\lambda\mu}\left(\partial_\beta g_{\mu\gamma} + \partial_\gamma g_{\mu\beta} - \partial_\mu g_{\beta\gamma}\right)$

Prolongations of the isometry PDR

Example

Its second prolongation I^2_g is the closed subset of $J^3(M^n, \mathbb{R}^q)$ $(x^\alpha, f^a, v^a_\alpha, v^a_{\alpha\beta}, v^a_{\alpha\beta\gamma})$ of codimension s_{n+1} defined by the system

$$\delta_{ab} v^a_\alpha v^b_\beta = g_{\alpha\beta}(x)$$

$$\delta_{ab} v^a_\alpha v^b_{\beta\gamma} = g_{\alpha\lambda}(x)\Gamma^\lambda_{\beta\gamma}(x)$$

$$\delta_{ab}\left(v^a_\alpha v^b_{\beta\gamma\mu} + v^a_{\alpha\mu} v^b_{\beta\gamma}\right) = \partial_\mu\left(g_{\alpha\lambda}(x)\Gamma^\lambda_{\beta\gamma}(x)\right)$$

Prolongations of the isometry PDR

Example

Its second prolongation $\mathcal{I}_g^2$ is the closed subset of $J^3(M^n, \mathbb{R}^q)$ $(x^\alpha, f^a, v^a_\alpha, v^a_{\alpha\beta}, v^a_{\alpha\beta\gamma})$ of codimension s_{n+1} defined by the system

$$\delta_{ab} v^a_\alpha v^b_\beta = g_{\alpha\beta}(x)$$

$$\delta_{ab} v^a_\alpha v^b_{\beta\gamma} = g_{\alpha\lambda}(x)\Gamma^\lambda_{\beta\gamma}(x)$$

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The last set of equations entails compatibility conditions

$$\delta_{ab}\left[v^a_{\alpha\gamma} v^b_{\beta\mu} - v^a_{\alpha\mu} v^b_{\beta\gamma}\right] = \partial_\mu\left[g_{\alpha\lambda}(x)\Gamma^\lambda_{\beta\gamma}(x)\right] - \partial_\beta\left[g_{\alpha\lambda}(x)\Gamma^\lambda_{\mu\gamma}(x)\right]$$

that are non-trivial for $n > 1$, so $\mathcal{I}_g^2$ is not fibered over the whole $\mathcal{I}_g^1$ (namely $\mathcal{I}_g$ is not stable!) unless $n = 1$.

The h-Principle

Definition

We say that a PDR $\mathcal{R}$ of order r satisfies the h-Principle (for C^r solutions) if every C^0 section $\varphi : E \to \mathcal{R}$ is homotopic to a holonomic section $j^r f$ by a continuous homotopy of sections $\varphi_t : E \to \mathcal{R}$, $t \in [0, 1]$.

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We say that a PDR $\mathcal{R}$ of order r satisfies the h-Principle for C^∞ (C^{an}) solutions if $\mathcal{R}^k$ is stable for some $k \geq 0$ and every C^0 section $\varphi : E \to \mathcal{R}^k$ is homotopic to a C^∞ (C^{an}) holonomic section $j^{r+k} f$ by a continuous homotopy of sections $\varphi_t : E \to \mathcal{R}^k$, $t \in [0,1]$.

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The idea of the h-Principle is that, "if there is enough space", *formal solutions* can be deformed into *true (holonomical) solutions*.
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Theorem (Grauert 1957)

Let G be a complex Lie group with a complex analytic subgroup H and consider a complex analytic fibration $F \to E$ with group structure G and fiber G/H.

Then, if V Stein, every continous section of $F \to E$ can be homotoped to a holomorphic one.

In Gromov's language, the Cauchy-Riemann PDR satisfies the C^{an} h-princple (note that this (closed) PDR is *stable*).

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Theorem (Hirsh 1959, 1961)

Let M, N be two smooth manifolds. Then every continuous section of the bundle $GL(TM, TN) \to M$ can be homotoped to the 1-jet of an immersion $M \to N$ in the following two cases:

- *Extra dimension:* $\dim N > \dim M$;
- *Critical Dimension:* $\dim N = \dim M$ *and N is open.*

In Gromov's language, the immersion PDR satisfies the C^∞ h-princple (note that this PDR is open and, therefore, *stable*).

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The idea of the h-Principle is that, "if there is enough space", *formal solutions* can be deformed into *true (holonomical) solutions*.
This idea arose out of a series of several discoveries in immersions theory:

Theorem (Nash 1954, Kuiper 1955)

Every C^1 immersion $M \to \mathbb{R}^q$ admits a C^1 homotopy of immersions to an isometric immersion.

In Gromov's language, the immersion PDR satisfies the C^1 h-Principle (note that this PDR is not stable so these solutions cannot be made more regular).

Techniques to prove the h-Principle

The last two theorems provide two of the main techniques used by Gromov to prove the validity of the h-Principle for a PDR:

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- *Sheaves technique,* based on the work of Smale and Hirsch;
- *Convex integration,* based on the work of Nash and Kuiper, especially useful in case of closed PDR;
- *Removal of Singularities technique,* developed by Gromov and Eliashberg, especially useful in the complex analytical or alebraic setting.

h-Principle & Free maps

Theorem

Let M^n be a smooth manifold and $\mathcal{F} \subset J^2(M, \mathbb{R}^q)$ the free maps PDR – namely in coordinates $(x^\alpha, f^a, v^a_\alpha, v^a_{\alpha\beta})$ every fiber of $\mathcal{F}$ is defined as rk $\left(v^a_\alpha, v^a_{\alpha\beta}\right) = s_n$.

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Then every continuous section of $\mathcal{F} \to M$ can be homotoped to the 2-jet of a free map $M \to \mathbb{R}^q$ in the following two cases:

- *Extra dimension: $q > n + s_n$;*
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Then every continuous section of $\mathcal{F} \to M$ can be homotoped to the 2-jet of a free map $M \to \mathbb{R}^q$ in the following two cases:

- *Extra dimension: $q > n + s_n$;*
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Corollary

If M is a stably parallelizable n-manifold (e.g. any orientable hypersurface of $\mathbb{R}^{n+1}$), it admits a free map into $\mathbb{R}^{n+s_n+1}$.

h-Principle & Free maps

Open Question

Do free maps of closed manifolds M, $\dim M \geq 2$, satisfy the h-Principle for $n + s_n$?

Does every parallelizable manifold (in particular the n-torus) admit a free map into $\mathbb{R}^{n+s_n}$?

h-Principle for C^k isometries, $k \geq 3$.

Although the isometries PDR $\mathcal{I}_g$ is not stable for any metric g, the PDR $(\mathcal{F} \cap \mathcal{I}^1_g)^1$ is stable for all metrics g.

Theorem

Let (M^n, g) be a C^k Riemannian manifold, $k \geq 5$. Then free isometric C^k immersions $M \to \mathbb{R}^q$ satisfy the h-Principle for $q \geq s_{n+1}$.

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Since $\mathcal{F} \to M$ admits always a section for $q \geq 2n + s_n$, this is equivalent to

Theorem

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