A quick survey of Hamiltonian systems

from a Dynamical Systems point of view

Roberto De Leo
Department of Mathematics

Plan of the presentation:

- General properties of Hamiltonian systems
  geodesics equations, Maupertuis principle, generating functions, Lagrange submanifolds, Hamilton-Jacobi, Huygens principle, relations between classical and quantum mechanics, optics, non-Hausdorff manifolds, Completely Integrable Systems, KAM

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- Systems with symmetries

  Noether's theorem, symplectic reduction, momentum map, Atiyah-Sternberg theorem

Main Sources

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Introduction

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3. $\phi_\xi^t \phi_\xi^s = \phi_\xi^{t+s}$ when all three maps are defined.

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Example: Riemannian Manifolds

Here h is symmetrical and ξ_H is known as the *gradient* of H

$$L_{\xi_H} H(x) \stackrel{\text{def}}{=} \frac{d}{dt} H(\phi^t_\xi(x))\Big|_{t=0}$$

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It turns out thought that the dynamics is much richer when h is antisymmetric.

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Locally we have coordinates $(q^1, \dots, q^n, p_1, \dots, p_n)$ where $\omega = dq^\alpha \wedge dp_\alpha$ (Darboux Theorem), so that

$$\xi_H = \frac{\partial H}{\partial p_\alpha}\frac{\partial}{\partial q^\alpha} - \frac{\partial H}{\partial q^\alpha}\frac{\partial}{\partial p_\alpha} = \frac{\partial H}{\partial p_\alpha}\partial_\alpha - \frac{\partial H}{\partial q^\alpha}\partial^\alpha$$

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This is the qualitative behviour of Hamiltonian orbits in 2 dimensions close to a stable equilibrium point.

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In case $V \equiv 0$, $m = 1$, we get the equation of geodesics:
$d/dt(g_{\alpha\beta} \dot{q}^\alpha) + \frac{1}{2} \partial_\alpha g^{\mu\nu} g_{\lambda\mu} g_{\rho\nu} \dot{q}^\lambda \dot{q}^\rho = 0$

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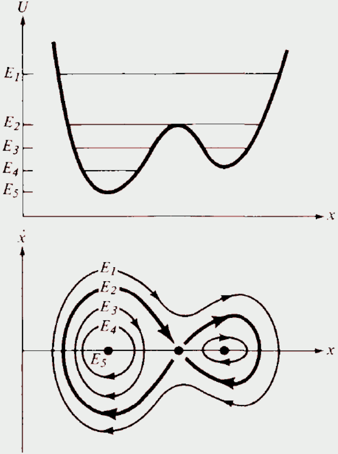

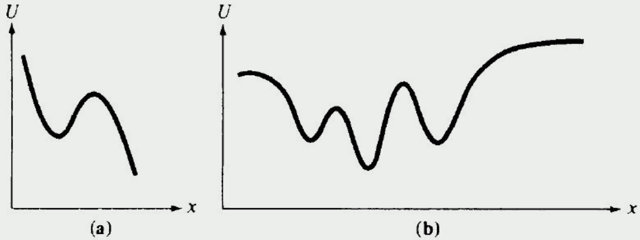

Figure 11 Potential energy

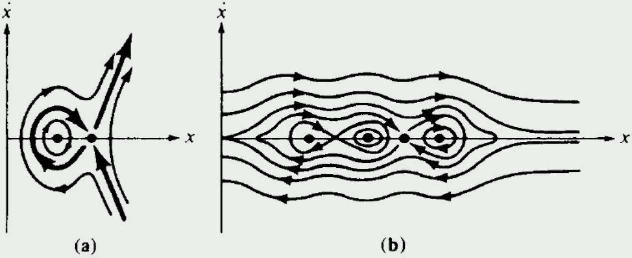

Figure 12 Phase curves

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Such systems have not been studied much so far, mainly because they do not arise from the framework of classical mechanics.

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A sign that ξ_η is not Hamiltonian is that its orbits, i.e. the level sets of H, are topologically non-trivial.

Example 2: smooth obstructions

Definition

A regular 1-dim. foliation $\mathcal{F}$ of $\mathbb{R}^2$ is Hamiltonian if its leaves are the level sets of a regular smooth function H, i.e. if $\mathcal{F} = \{dH = 0\}$, i.e. if $T_x\mathcal{F} = \mathrm{span}\{\xi_H(x)\}$ for all $x \in \mathbb{R}^2$.

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Remark: $\mathcal{F}$ is usually a non-Hausdorff space, but this is not an obstruction to define a smooth structure[1].

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Of course, though, fundamental properties such as the existence of a partition of unity do not hold in non-Hausdorff spaces!

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This is the exception rather than the rule. It turns out, for example, that there exist foliations such that $C^1(\mathcal{F})$ contains only constant functions! (see the article by Haefliger and Reeb and the references therein).

Locally every regular foliation is Hamiltonian but globally things are different:

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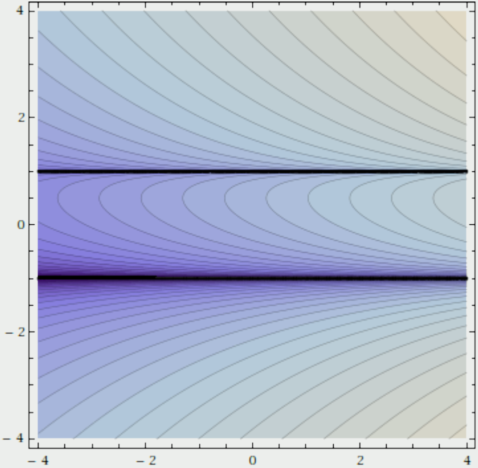
This is the exception rather than the rule. It turns out, for example, that there exist foliations such that $C^1(\mathcal{F})$ contains only constant functions! (see the article by Haefliger and Reeb and the references therein).
Incidentally, we have an interesting related property:

Theorem (Haefliger, Reeb 1957)

On non-Hausdorff smooth manifolds of every dimension there are infinitely many inequivalent smooth structures.

Example: a non-Hamiltonian foliation of $\mathbb{R}^2$

Consider $(\mathbb{R}^2, dq \wedge dp)$ and $\eta = (1-p^2)dq + 2(1-2p)dp$.
Its leaves are shown below:



Clearly $\mathcal{F}_\eta = \{\eta = 0\}$ is a regular foliation but no regular function has this foliation as the set of its level curves.

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Correspondigly, the vector field $\xi_\eta = 2(2p-1)\partial_q + (1-p^2)\partial_p$ is regular and everywhere tangent to $\mathcal{F}_\eta$ but $\ker L_{\xi_\eta}$ is generated by

$$H(q,p) = (p+1)^3(p-1)e^q$$

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whose differential vanishes on the leaf $p = -1$.

Hence the derivative of every function of

$$C^1(\mathcal{F}_\eta) = \{f \circ H \,|\, f \in C^1(\mathbb{R})\}$$

is null in that point.

In coordinates, consider on

$$\mathcal{F}_\eta \simeq Y = \mathbb{R} \sqcup \mathbb{R}/\{x \sim y \text{ if } x = y \text{ and } x < 0\}$$

the two charts $\psi, \phi : (-\varepsilon, \varepsilon) \to Y$ s.t.
$\psi(w)$ is the leaf of η passing through $(0, -1 - w)$ and
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Since w and z are the coords of the same leaf iff $H(0, -1 - w) = H(0, z + 1)$, the coords change is given by

$$w^3(1 + w) = z(z + 2)^3$$

which reduces to $z \simeq w^3$ close enough to 0.

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Given $f \in C^1(\mathcal{F}_\eta)$, then its representatives in coordinates are
$f_\psi(w) = f(\psi(w))$ and $f_\phi(z) = f(\phi(z))$. Then

$$f_\psi(w) = f_\phi \circ \phi^{-1} \circ \psi(w) = f_\phi(w^3)$$

and

$$f'_\psi(w)|_{w=0} = 3w^2 f_\phi(w^3)|_{w=0} = 0.$$

While in the example of the torus the vector field was only locally Hamiltonian for *topological* (C^0) reasons, here it depends on the *smooth* (C^1) structure:

Theorem (DL, 2014)

There exists a continuous funtion G such that (H, G) *is locally injective and* $\mathcal{F}_\eta$ *is Hamiltonian with respect to the (inequivalent) smooth structure on the plane given by the charts* (H, G) *at every point.*

Related Literature

• S.P. Novikov, *The Hamiltonian formalism and a many-valued analogue of Morse theory*, Uspekhi Mat. Nauk , 1982, 37:5, 3-49.

• S.P. Novikov, *The Semiclassical Electron in a Magnetic Field and Lattice. Some Problems of the Low Dimensional Periodic Topology*, Geometric and Functional Analysis, 1995, 5:2, 434-444.

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• R. De Leo, *Solvability of the cohomological equation for regular vector fields on the plane*, Annals of Global Analysis and Geometry, 2011, 39:3, 231-248

Least Action Principles

The Poincaré-Cartan 1-form $\theta_H = p_\alpha dq^\alpha - Hdt$

Recall that the trajectory of a Hamiltonian system on M starting at time t_0 from q_0 and arriving at time t_1 in q_1 is an extremal of the action

$$S = \int_\gamma L(q, \dot{q}) dt = \int_\gamma (p_\alpha dq^\alpha - Hdt),$$

$$\gamma \in \{\gamma : [t_0, t_1] \to M \,|\, \gamma(t_0) = q_0, \gamma(t_1) = q_1\}$$

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The **Poincaré-Cartan 1-form**

$$\theta_H(t,q,p) = p_\alpha dq^\alpha - H(p,q)dt \in \Omega^1(\mathbb{R} \times T^*M)$$

plays a fundamental role in Hamiltonian systems.

Least Action principle in $\mathbb{R} \times T^*M$

Theorem (see Arnold, 45C)

The extremals of the "extended action"

$$S_{\mathbb{R}\times T^*M}[\gamma] = \int_\gamma \theta_H$$

in the space of all paths $\gamma : [t_0, t_1] \to \mathbb{R} \times T^*M$ *such that*

$\pi_t(\gamma(t)) = t$, $\pi_M(\gamma(t_0)) = (t_0, q_0)$ *and* $\pi_M(\gamma(t_1)) = (t_1, q_1)$,

where $\pi_t(t, q, p) = t$ *and* $\pi_M(t, q, p) = (t, q)$,

are the solutions $\gamma = (t, q(t), p(t)) : [t_0, t_1] \to \mathbb{R} \times T^*M$

of the Hamilton equations satisfying the initial conditions

$q(t_0) = q_0$, $q(t_1) = q_1$.

Remark: no condition is put on $p(t_0), p(t_1)$!

Proof.

We consider a family of paths γ_ε and set $\delta = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}$. Then

$$\delta \int_{\gamma_\varepsilon} \theta_H = \int_\gamma [p_\alpha \delta \dot{q}^\alpha + \dot{q}^\alpha \delta p_\alpha - \partial_\alpha H \delta q^\alpha - \partial^\alpha H \delta p_\alpha]\, dt =$$

$$= p_\alpha \delta q^\alpha \Big|_{t_0}^{t_1} + \int_\gamma [(\dot{q}^\alpha - \partial^\alpha H)\delta p_\alpha + (-\dot{p}_\alpha + \partial_\alpha H)\delta q^\alpha]\, dt$$

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From the line above it is clear why we need to fix the initial conditions for the q (i.e. $\delta q = 0$ at t_0 and t_1) but <u>not</u> for the p.

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It looks surprising that the extremals of the action on M coincide with those of the corresponding action on $\mathbb{R}\times T^*M$, where the p_α are allowed to vary independently from the q^α.

The reason behind this is that, for fixed $\dot q^\alpha$ on TM, the value of $p_\alpha = \frac{\partial L(q,\dot q)}{\partial\dot q^\alpha}$ is, by definition of Legendre transform, an extremal of the function $L = p_\alpha\dot q^\alpha - H$.

Least Action Principle in $M_E = H^{-1}(E)$
Maupertuis Principle, Hamiltonian version

Theorem (Mapertuis principle I, see DFN Thm33.3.1)

The extremals of the "truncated action"

$$S_E[\gamma] = \int_\gamma \theta, \quad \theta = p_\alpha dq^\alpha \text{ (Liouville 1-form)},$$

in the space Ω *of all paths* $\gamma : [t_0, t_1] \to T^*M$ *such that*

$\pi_M(\gamma(t_0)) = q_0, \pi_M(\gamma(t_1)) = q_1, \gamma([t_0, t_1]) \subset M_E,$

where $\pi_M : T^*M \to M$ *is the projection that "drops" the p,*

are all the reparametrizations of the solutions $\gamma : [t_0, t_1] \to T^*M$

of the Hamilton equations contained inside Ω*.*

Proof.

Proceeding as in the previous case, we find that

$$\delta \int_{\gamma_\varepsilon} \theta = \int_\gamma [\dot{q}^\alpha \delta p_\alpha - \dot{p}_\alpha \delta q^\alpha]\, dt.$$

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This time though the δq and δp are not independent: since H is constant over all paths, then

$$0 = \delta [H(q_\varepsilon(t), p_\varepsilon(t))] = \partial_\alpha H \delta q^\alpha + \partial^\alpha H \delta p_\alpha$$

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$$0 = \delta[H(q_\varepsilon(t), p_\varepsilon(t))] = \partial_\alpha H \delta q^\alpha + \partial^\alpha H \delta p_\alpha$$

Since this is the only constraint, it means that

$$(\dot{q}^\alpha, \dot{p}_\alpha) \propto (\partial^\alpha H, -\partial_\alpha H),$$

namely the paths γ that extremizes the truncated action are those whose image $\gamma(M) \subset M_E$ coincides with the image of a solution of the coresponding Hamiltonian equations of motions, i.e. is a solution modulo reparametrization. □

Least Action Principle in $M_E = H^{-1}(E)$
Maupertuis Principle, Lagrangian version

Theorem (Mapertuis pr. II, Arn 45D & AM Thm3.8.5)

Consider a Hamiltonian system H with Lagrangian $L(q,\dot{q}) = \dot{q}\partial L/\partial\dot{q} - H(q,\partial L/\partial\dot{q})$.

Among all curves $\gamma = q(t) : \mathbb{R} \to M$ *connecting* $q_0, q_1 \in M$ *and parametrized so that* $H(q,\partial L/\partial\dot{q}) = E$*, the extremals of the "truncated action"*

$$S_E[\gamma] = \int_\gamma \theta = \int_{t_0}^{t_1} \frac{\partial L}{\partial \dot{q}^\alpha}\dot{q}^\alpha dt\,,$$

are all reparametrizations of the solutions of the Lagrangian equations of motion which keep the energy equal to E.

Proof.

Let $\mathcal{L} : TM \to T^*M$ be the Legendre transformation and consider any curve $\gamma = q(t) : \mathbb{R} \to M$ connecting q_0 with q_1 in such a way that $H(q(t), \partial L/\partial\dot{q}) = E$.

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Then the curve $\tilde{\gamma} = \mathcal{L} \circ \gamma : \mathbb{R} \to T^*M$ satisfies the conditions of the Maupertuis' principle in the Hamiltonian version

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and therefore it is an extremal of the truncated action

iff $\tilde{\gamma}$ is a reparametrization of the solutions of the Hamiltonian equations of motion

iff γ is a reparametrization of the solutions of the Lagrangian equations of motion. □

Example 1: Geodesics

Theorem

On a Riemannian manifold (M, g)*, the extremals of the action* $S = \int_\gamma \sqrt{g_{\alpha\beta}\dot{q}^\alpha \dot{q}^\beta}dt$ *are (unparametrized) geodesics.*

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Proof.

Geodesics are the solutions of the Hamiltonian dynamical system given by the purely kinetic energy Hamiltonian $H(q,p)=\frac{1}{2}g^{\alpha\beta}p_\alpha p_\beta$. On $H=E$, $g^{\alpha\beta}p_\alpha p_\beta=\sqrt{E}\sqrt{g^{\alpha\beta}p_\alpha p_\beta}$ and so the extremals of $\int_\gamma g_{\alpha\beta}\dot q^\alpha\dot q^\beta dt$ are also extremals of $\int_\gamma\sqrt{g_{\alpha\beta}\dot q^\alpha\dot q^\beta}dt$. □

Example 2: Motion in a Riemann Manifold

Theorem

A particle of mass m on a Riemannian manifold (M, g) *subjected to a potential* $V(q)$ *moves, at the energy level* E*, along the geodesics of the new metric*

$$\tilde{g}_{\alpha\beta} = 2m(E - V(x))g_{\alpha\beta}\,.$$

Example 2: Motion in a Riemann Manifold

Proof.

If $H(q,p) = \frac{1}{2}g^{\alpha\beta}p_\alpha p_\beta + V(q) = \frac{1}{2}g_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta + V(q)$, then, in M_E,

$$g_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta = 2(E - V(q)).$$

Hence

$$S[\gamma] = \int_\gamma p_\alpha dq^\alpha = \int_\gamma g_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta dt\,,$$

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Hence

$$S[\gamma] = \int_\gamma p_\alpha dq^\alpha = \int_\gamma g_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta dt\,,$$

which we can write as

$$S[\gamma] = \int_\gamma \sqrt{2(E - V(q))}\sqrt{g_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta}dt = \int_\gamma \sqrt{\tilde{g}_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta}dt\,,$$

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from which it is clear that the extremals of the Maupertuis action with energy E coincide with the geodesics of M with respect to the metric

$$\tilde{g}_{\alpha\beta} = 2(E - V(q))g_{\alpha\beta}.$$

Maupertuis' principle allows us to apply to Hamiltonian dynamics important results of Riemannian geometry, e.g. the fact that, if in some homotopy class of loops there is a curve of shortest length, this is a geodesics:

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Double Pendulum

Corollary (See Arn 45C)

For every n_1, n_2 there is a periodic motion of the double pendulum ($M = \mathbb{T}^2$) such that one pendulum makes n_1 oscillations while the other makes n_2 oscillations.

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Rigid Body

Corollary (See Arn 45C)

Given a rigid body ($M = SO_3$), in any potential field there exists at least one periodic motion of the body. Moreveor, there are periodic motions for every arbitrary high value of the energy.

Example: Motion of light

The Hamiltonian for rays of light is $H(q,p) = c(q)\|p\|$.

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Theorem (Fermat's principle – Novikov 33.3.3)

The path that light rays take by passing from a point A to a point B in a isotropic media are geodesics with respect to the metric $g_{\alpha\beta} = \frac{1}{c(q)^2}\delta_{\alpha\beta}$.

Related Literature

• S.P. Novikov, *The Hamiltonian formalism and a many-valued analogue of Morse theory*, RMS , 1982, 37:5, 3-49.

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Hamiltonian Systems as Lagrangian Submanifolds

Generating Functions

Symplectic diffeomorphisms of a manifold M^{2n}, which are $2n$ maps of $2n$ variables, are actually determined by a single function of $2n$ variables:

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Theorem

$f:(M^{2n},\omega_1)\to(N^{2n},\omega_2)$ is symplectic iff f's graph $\Gamma_f\subset M\times N$ is Lagrangian submanifold of $(M\times N,\omega_1-\omega_2)$.

Definition

Let $\theta_{1,2}$ be local Liouville 1-forms for $\omega_{1,2}$ and $i:\Gamma_f\to M\times N$ the inclusion of the graph. Then locally $i^*(\theta_1-\theta_2)=dS$.
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S is the *generating function* for f.

This means that locally $\theta_1 - \theta_2 = p_\alpha dq^\alpha - P_a dQ^a = dS(q, Q)$, i.e. locally

$$p_\alpha = \frac{\partial S}{\partial q^\alpha}, \; P_a = \frac{\partial S}{\partial Q^a}.$$

Hamiltonian and Lagrangian formulations via Lagrangian submanifolds

Lagrangian submanifolds are a powerful language in the framework of Hamiltonian dynamics. In particular we can reformulate the whole theory with this language:

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Consider the following symplectic bundles and diffeomorphisms:

T^*M	$T^*(T^*M)$	$T(T^*M)$	$T^*(TM)$
(q^α, p_α)	$((q^\alpha, p_\alpha), (w_\alpha, v^\alpha))$	$((q^\alpha, p_\alpha), (v^\alpha, w_\alpha))$	$((q^\alpha, v^\alpha), (p_\alpha, w_\alpha))$
$p_\alpha dq^\alpha$	$w_\alpha dq^\alpha + v^\alpha dp_\alpha$	$v^\alpha dp_\alpha - w_\alpha dq^\alpha$	$p_\alpha dq^\alpha + w_\alpha dv^\alpha$

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$p_\alpha dq^\alpha$	$w_\alpha dq^\alpha + v^\alpha dp_\alpha$	$v^\alpha dp_\alpha - w_\alpha dq^\alpha$	$p_\alpha dq^\alpha + w_\alpha dv^\alpha$

$$\psi : \begin{array}{ccc} T(T^*M) & \to & T^*(T^*M) \\ ((q^\alpha, p_\alpha), (v^\alpha, w_\alpha)) & \mapsto & ((q^\alpha, p_\alpha), (-w_\alpha, v^\alpha)) \end{array}$$

Hamiltonian and Lagrangian formulations via Lagrangian submanifolds

Lagrangian submanifolds are a powerful language in the framework of Hamiltonian dynamics. In particular we can reformulate the whole theory with this language:

Consider the following symplectic bundles and diffeomorphisms:

T^*M	$T^*(T^*M)$	$T(T^*M)$	$T^*(TM)$
(q^α, p_α)	$((q^\alpha, p_\alpha), (w_\alpha, v^\alpha))$	$((q^\alpha, p_\alpha), (v^\alpha, w_\alpha))$	$((q^\alpha, v^\alpha), (p_\alpha, w_\alpha))$
$p_\alpha dq^\alpha$	$w_\alpha dq^\alpha + v^\alpha dp_\alpha$	$v^\alpha dp_\alpha - w_\alpha dq^\alpha$	$p_\alpha dq^\alpha + w_\alpha dv^\alpha$

$$\psi : \begin{array}{ccc} T(T^*M) & \to & T^*(T^*M) \\ ((q^\alpha, p_\alpha), (v^\alpha, w_\alpha)) & \mapsto & ((q^\alpha, p_\alpha), (-w_\alpha, v^\alpha)) \end{array}$$

$$\phi : \begin{array}{ccc} T(T^*M) & \to & T^*(TM) \\ ((q^\alpha, p_\alpha), (v^\alpha, w_\alpha)) & \mapsto & ((q^\alpha, v^\alpha), (w_\alpha, p_\alpha)) \end{array}$$

The pull-back $\psi^*\theta_{T^*(T^*M)}$, $\phi^*_{T^*(TM)}\theta_2$ on $T(T^*M)$ of the canonical Liouville 1-forms on $T^*(T^*M)$ and $T^*(TM)$ are given by:

$$\psi^*\theta_{T^*(T^*M)} = v^\alpha dp_\alpha - w_\alpha dq^\alpha , \quad \phi^*\theta_{T^*(TM)} = p^\alpha dv_\alpha + w_\alpha dq^\alpha$$

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Theorem (Tulczyjew 1974)

Consider the symplectic structure $\omega = dv^\alpha \wedge dp_\alpha - dw_\alpha \wedge dq^\alpha$ *on* $T(T^*M)$*. Then:*

1. ψ *is symplectic,* ϕ *is anti-symplectic;*

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4. $H(q,p) = p_\alpha v^\alpha - L(q,v)$, *with* $p_\alpha = \dfrac{\partial L}{\partial v^\alpha}$.

Related Literature

- W.M. Tulczyjew *Hamiltonian Systems, Lagrangian systems and the Legendre transformation*, Symp. Math. 14, 247-258
- K. Konieczna, P. Urbanski *Double vector bundles and duality*, Archivum Mathematicum, 1999
- M. de Leon, D. Martin de Diego, *Tulczyjew's triples and lagrangian submanifolds in classical field theories*, in "Applied Differential Geometry and Mechanics,", Eds. W. Sarlet and F. Cantrijn, U. of Gent, Gent, Academia Press (2003), 21-47.
- J. Grabowski, G. Marmo, *Deformed Tulczyjew triples and Legendre transform*, Rend. Sem. Univ. Pol. Torino, 1999, 54:3

Hamilton-Jacobi Equation

Huygens principle

The idea behind Hamilton-Jacobi equations comes from the Huygens principle in optics:

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Consider the light emanating from a point q_0. The wave front $\Phi_{q_0}(t+s)$ is the envelope of the fronts $\Phi_q(s)$ for all $q \in \Phi_{q_0}(t)$.

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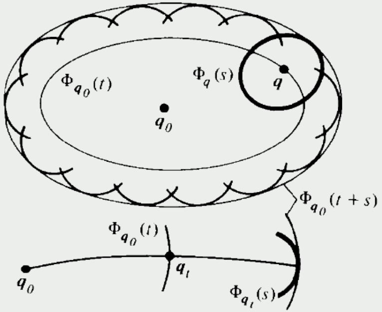

Figure 193 Envelope of wave fronts

Huygens principle

The level set of $S_{q_0}(q)$ (optical length) is the wave front.

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Note that the directions of $\dot{q}$ and p do not coincide in an anisotropic medium!

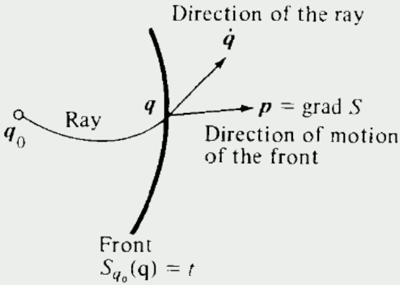

Figure 195 Direction of a ray and direction of motion of the wave front

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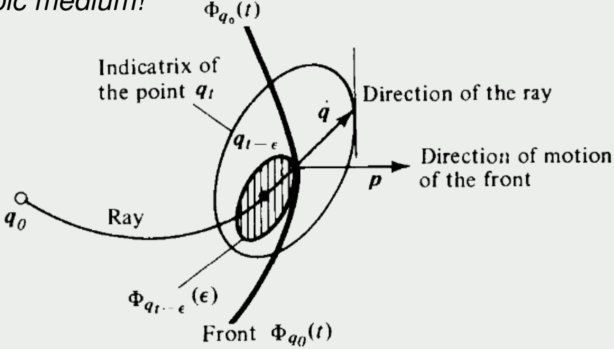

Figure 197 Conjugacy of the direction of a wave and of the front

Optical-Mechanics Analogy

Optics	Mechanics
Optical medium	Extended configuration space $\{(\mathbf{q}, t)\}$
Fermat's principle	Hamilton's principle $\delta \int L\, dt = 0$
Rays	Trajectories $\mathbf{q}(t)$
Indicatrices	Lagrangian L
Normal slowness vector $\mathbf{p}$ of the front	Momentum $\mathbf{p}$
Expression of $\mathbf{p}$ in terms of the velocity of the ray, $\dot{\mathbf{q}}$	Legendre transformation
1-form $\mathbf{p}\, d\mathbf{q}$	1-form $\mathbf{p}\, d\mathbf{q} - H\, dt$

Hamilton-Jacobi equations v1

The connection between Huygens principle and Hamiltonian equations comes from the three following observations:

Theorem 1

The 1-form $\eta \in \Omega^1(M)$ *is closed iff* $\eta^*\omega = 0$*, i.e. iff its graph* $\eta(M) \subset T^*M$ *is a Lagrangian submanifold of* T^*M*.*

Proof.

$$\eta^*\omega = dq^\alpha \wedge d\eta_\alpha = \partial_\beta \eta_\alpha dq^\alpha \wedge dq^\beta =$$
$$= \frac{1}{2}(\partial_\beta \eta_\alpha - \partial_\alpha \eta_\beta) dq^\alpha \wedge dq^\beta$$ □

Hence locally $\alpha = dS$, namely $\alpha(M)$ writes as $p_\alpha = \dfrac{\partial S}{\partial q^\alpha}$.

Theorem 2

Let $\Gamma^n \subset T^*M^n$ be Lagrangian and contained in $H^{-1}(E_0)$. Then $\xi_H \in T\Gamma$.

Proof.

Since $\omega(\xi_H, \zeta) = dH(\zeta) = 0$, $\forall \zeta \in T\Gamma$, and Γ is Lagrangian, then $\xi_H \in T\Gamma$ at every point. □

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Proof.

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Theorem 3

Let $\Gamma^{n-1} \subset T^*M^n$ be isotropic. Then

$$\Gamma^n_T = \bigcup_{t \in [0,T]} \phi^t_H(\Gamma^{n-1})$$

is Lagrangian $\forall T > 0$.

Theorem (HJ v1, DFN 35.1.6, AM 5.2.18)

Given a Hamiltonian H on T^*M and a closed 1-form η on M, the following are equivalent:

1. $d(\eta^* H) = 0$;

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3. *for every curve* $\gamma = q(t) : \mathbb{R} \to M$ *satisfying* $\dot{q}^\alpha = \left.\dfrac{\partial H}{\partial p_\alpha}\right|_{\eta(q)}$, *the curve* $\tilde{\gamma}(t) = \eta(q(t))$ *is an integral curve of* ξ_H;

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4. *if S is a generating function for* $\eta(M)$, *namely if locally* $\eta = dS$, *then S satisfies the (time-independent) Hamilton-Jacobi equation*

$$H\left(q, \frac{\partial S}{\partial q}\right) = E_0$$

The name S for the generating function was not by chance:

Theorem

Let $\Gamma \subset T^*M$ be Lagrangian and contained in $H = E_0$, $m_0, m \in \Gamma$ two "close enough" points and $\gamma_{1,2} : [0,1] \to \Gamma$ two paths s.t. $\gamma_{1,2}(0) = m_0$ and $\gamma_{1,2}(1) = m$. Then $\int_{\gamma_1} \theta = \int_{\gamma_2} \theta$.

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Proof.

Since Γ is Lagrangian,

$$d\theta|_\Gamma = \omega|_\Gamma = 0$$

and so locally $\theta = dS$, i.e.

$$p_\alpha = \frac{\partial S}{\partial q^\alpha},$$

and therefore

$$\int_{\gamma_i} \theta = S(m) - S(m_0).$$

Corollary ("Method of Characteristics")

For a fixed q_0, assume that the Lagrangian submanifold $\Gamma^n \subset \{H(q,p) = E_0\} \subset T^*M$ projects with full rank on M close to q_0. Then the "truncated action"

$$S_{E_0}(q) = \int_{q_0}^{q} p_\alpha dq^\alpha$$

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Proof.

Since $dS_{E_0} = p_\alpha dq^\alpha$, we have that

$$p_\alpha = \frac{\partial S_{E_0}}{\partial q^\alpha}$$

so that, since $\Gamma \subset \{H(q,p) = E_0\}$, $H(q, \partial_q S_{E_0}) = E_0$. □

Application to solving 1st order PDEs

Consider the 1st order implicit PDE with "Cauchy boundary conditions":

$$H(q, \partial_q S) = E_0 \;,\;\; S\Big|_{\Gamma^{n-1}} = s_0 \in C^\infty(\Gamma^{n-1})$$

where $H|_{\Gamma^{n-1}} = E_0$, Γ^{n-1} is transversal to the Hamiltonian flow of H and projects diffeomorphically on M.

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Then the previous Corollary shows that, at least for small T, the solution on $\pi_M(\Gamma_T)$ is given by

$$S_{E_0}(q) = s_0(q_0) + \int_{q_0}^{q} p_\alpha dq^\alpha \,,$$

where q_0 is the point of Γ^{n-1} such that $q = \Phi^t_H(q_0)$ for some t.

Example 1: Harmonic Oscillator

$$H(x,y,p_x,p_y) = \frac{1}{2}\left(p_x^2 + p_y^2 + x^2 + y^2\right)$$

The level set $H = \frac{1}{2}$ is the unitary 3-sphere $\mathbb{S}^3$.

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Every orbit is periodic with period 2π and lies on a torus $p_x^2 + x^2 = \alpha^2$, $p_y^2 + y^2 = 1 - \alpha^2$, so the manifold of trajectories $\Gamma_{2\pi}$ of every loop $\Gamma^1 \subset \mathbb{S}^3$ transversal to the flow is a 2-torus.

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$$x = \cos\phi\cos t\,, \quad y = \sin\phi\cos t$$

$$p_x = -\cos\phi\sin t\,, \quad p_y = -\sin\phi\sin t$$

Hence

$$S(x(T),y(T)) = \int_0^T (p_x\,dx + p_y\,dy) = \int_0^T \cos^2 t\,dt = \frac{1}{2}(T+\sin(2T))$$

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At time T, $x(T) = r(T)\cos\phi$, namely $r(T) = \cos^{-1} T$, so

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Remarks: 1. In order to have a solution on the whole $r \leq 1$, we must have $s_0 = const$.
2. The solution is singular where Γ_t is not a graph.

Example 2: Cohomological Equation

$$L_\xi f = g\,,\ \xi \in \chi(M)\,,\ f, g \in C^\infty(M)$$

Every $\xi \in \chi(M)$ is the base component of a Ham. vector field ξ_H: just take $H(q,p) = p_\alpha \xi^\alpha(q)$.

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$$L_\xi f = g$$

with the Method of Characteristics consider the Hamiltonian

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Once the value of f is given on some $n-1$-dimensional submanifold transversal to ξ, its (local) solution is given by

$$f(q) = \int_{q_0}^{q} p_\alpha dq^\alpha = \int_{t_0}^{t} \frac{\partial f}{\partial q} \dot{q}^\alpha dt = \int_{t_0}^{t} \frac{\partial f}{\partial q} \xi^\alpha dt = \int_{t_0}^{t} g(q(t))dt\,,$$

(the integral is taken over the integral traj. of ξ joining q_0 and q)

Hamilton-Jacobi equations v2

An alternate way to look at the HJ equation is that we want to find a symplectic diffeomorphism $\psi : (q,p) \to (Q,P)$ where the Hamiltonian writes in a simpler way.

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This new generating function S therefore satisfies the HJ eq. but it also depends on n "external parameters" Q^α, so that it gives rise to a Lagrangian foliation of T^*Q where every leaf is isoenergetic.

Hamilton-Jacobi equation (time-dependent)

In case of time-dependent Hamiltonians $H(t,q,p)$ we can repeat verbatim all we did so far using the following dictionary:

	time ind.	time dep.
base space	M	$\mathbb{R} \times M$

Hamilton-Jacobi equation (time-dependent)

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Ham. eqs.	$\begin{pmatrix}\dot q\\ \dot p\end{pmatrix} = \begin{pmatrix}\partial H/\partial p\\ -\partial H/\partial q\end{pmatrix}$	$\begin{pmatrix}\dot q\\ \dot p\\ \dot t\\ \dot E\end{pmatrix} = \begin{pmatrix}\partial H/\partial p\\ -\partial H/\partial q\\ 1\\ \partial H/\partial t\end{pmatrix}$

Hamilton-Jacobi equation (time-dependent)

In this environment, the generating function is given by

$$S(m) = S(m_0) + \int_\gamma [p_\alpha dq^\alpha - H(t,q,p)]\, dt$$

and satisfies the complete Hamilton-Jacobi equation

$$H\left(q, \frac{\partial S}{\partial q}\right) = -\frac{\partial S}{\partial t}$$

The solution to this equation provides a 1-parameter family of symplectomorphisms S_t which make the Hamiltonian H equal to constant at all time.

HJ equation and Quantum Mechanics

Feynmans' two postulates for QM on $\mathbb{R}^n$:

1. The probability $\langle q_1|\psi_t|q_2\rangle$ that a particle represented by the wavefunction $\psi_t \in L^2(\mathbb{R}^n)$ moves from q_1 to q_2 under a Hamiltonian $H(q,p) = \frac{1}{2m}\delta^{ij}p_ip_j + V(q)$ is the "sum" over all contribution from all possible paths joining the two points;

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$$\frac{\partial S}{\partial t} = \frac{1}{2m}\left[\delta^{ij}\frac{\partial S}{\partial q^\alpha}\frac{\partial S}{\partial q^\beta}\right] + V(q)$$

writes as

$$-i\hbar\dot\psi = -\frac{\hbar^2}{2m}\Delta\psi + V(q)\psi + \frac{i\hbar}{2m}\psi\Delta S$$

Apart for the non-linear term, this is exactly the Schrodinger equation of quantum mechanics $-i\hbar\dot\psi = \hat{H}\psi$, where $\hat{H}$ is comes from H via $p_\alpha \to -i\hbar\frac{\partial}{\partial q^\alpha}$ and $q^\alpha \to$ “multiplication by q^α”.

Now consider instead the Schrodinger equation

$$\frac{\hbar}{i}\dot{\psi} = -\frac{\hbar^2}{2m}\Delta\psi + V(q)\psi$$

and write $\psi_t(q) = e^{\frac{i}{\hbar}S[\gamma]}$. Which eq. does S satisfy?

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Proceeding like above we find

$$-\frac{\partial S}{\partial t} = \frac{1}{2m}\left[\delta^{ij}\frac{\partial S}{\partial q^i}\frac{\partial S}{\partial q^j}\right] + V(q) - \frac{i\hbar}{2m}\Delta S$$

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that, for $\hbar \to 0$, reduces exactly to the HJ equation!

This is the simplest way to show that QM reduces to CM for $\hbar \to 0$.

The WKB Method

Consider again the Schrodinger equation in $\mathbb{R}^n$

$$-i\hbar\dot{\psi} = -\frac{\hbar^2}{2m}\Delta\psi + V(q)\psi$$

Under the ansatz $\psi(x) = e^{iS(x)/\hbar}$, at 1st order in $\hbar$ then S is the solution of the corresponding HJ equation.
This though is a very poor approximation, e.g. $\psi \notin L^2(\mathbb{R}^n)$.
Under the ansatz

$$\psi(x) = a(x)e^{iS(x)/\hbar}$$

ψ is an eigenfunction for the quantum Hamiltonian $\hat{H}$ iff

$$i\hbar\left(a\Delta S + 2\delta^{\alpha\beta}\partial_\beta a\partial_\alpha S\right) + \hbar^2\Delta a = 0$$

At the 1st order in $\hbar$ we get the *homogeneous transport equation*

$$a\Delta S + 2\delta^{\alpha\beta}\partial_\alpha a\partial_\beta S = 0\,.$$

Example: QM on the line

The 2nd order solution $\psi = ae^{iS/\hbar}$ is called *semiclassical approximation* of the exact solution of the Schrodinger equation. In $\mathbb{R}$, the homegenous transport equation writes

$$aS'' + 2a'S' = 0$$

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This method, called WKB (Wentzel, Kramers, Brillouin), is at the base of *microlocal analysis*.

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• J. Carinena, X. Gracia, G. Marmo, E. Martinez, M Munoz-Lecanda, N. Roman-Roy, *Geometric Hamilton-Jacobi theory*, Int. J. Geom. Meth. Mod. Phys., 2006, 3, 1417-1458

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Completely Integrable Systems

Poisson bracket & First Integrals

Let (M^{2n}, ω) be a symplectic mfd. Functions in $C^\infty(M)$ are called *observables*.

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Theorem

If $\{f_1, \dots, f_n\}$ are n commuting observables in involution, all level submanifolds $f_1 = c_1, \dots, f_n = c_n$ are Lagrangian.

Arnold-Liouville Theorem

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2. *in the neighborhood of each such torus, there exists* action-angle *symplectic coordinates* $I_1, \dots, I_n, \varphi^1, \dots, \varphi^n$ *such that the* φ^α *are coordinates on the torus and* $H = H(I_1, \dots, I_n)$*.*

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In particular in such coordinates the Hamilton eqs writes

$$\dot{I}_\alpha = 0\,,\;\; \dot{\varphi}^\alpha = \frac{\partial H}{\partial I_\alpha}$$

Hamilton-Jacobi and CIS

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Theorem (Jacobi, see Arn 47B)

If the Hamilton-Jacobi equation $H(q, \partial_q S) = E_0$ admits a solution $S(q, Q)$, depending on n parameters $Q^1, \dots, Q^n$, such that the Hessian

$$\frac{\partial^2 S}{\partial q \partial Q}$$

is always non-degenerate, then the corresponding Hamiltonian equations

$$\begin{pmatrix} \dot q \\ \dot p \end{pmatrix} = \begin{pmatrix} \partial H/\partial p \\ -\partial H/\partial q \end{pmatrix}$$

can be solved explicitly by quadratures and the n functions $Q^\alpha(q, p)$ are all integrals of motion.

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The coord. change $(q,p) \mapsto (\varphi, I)$ is symplectic (i.e. $d\varphi \wedge dI = dq \wedge dp$) and the equations of motion now write

$$\dot{\varphi} = \omega, \quad \dot{I} = 0$$

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is an ellipsoid, elliptic hyperboloid and hyperbolic hyperboloid for respectively $i=1,2,3$.

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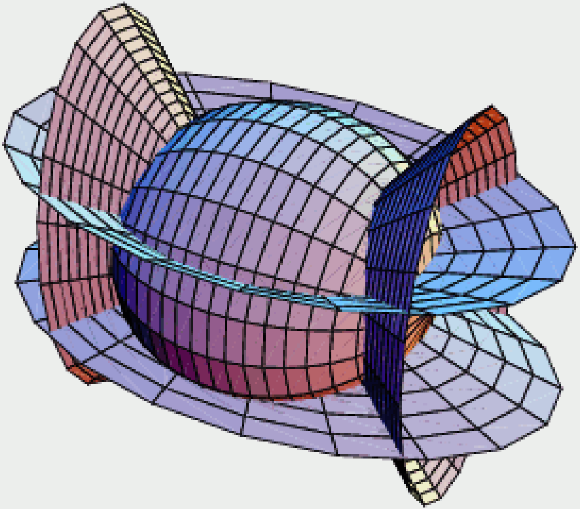
is an ellipsoid, elliptic hyperboloid and hyperbolic hyperboloid for respectively $i=1,2,3$.
Note that λ_i, λ_j can be used as local coordinates on E_k.



In coordinates λ_2, λ_3 on the ellipsoid E_1 the metric is

$$g = (\lambda_3 - \lambda_2)\left[\frac{\lambda_3 - \lambda_1}{f(\lambda_3)} d\lambda_3^2 - \frac{\lambda_2 - \lambda_1}{f(\lambda_2)} d\lambda_2^2\right],$$

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$$\text{So} \quad H(\lambda_2, \lambda_3) = \frac{1}{\lambda_3 - \lambda_2}\left[\frac{f(\lambda_3)}{\lambda_3 - \lambda_1} p_3^2 + \frac{f(\lambda_2)}{\lambda_2 - \lambda_1} p_2^2\right]$$

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and therefore the Hamilton-Jacobi equation

$$\frac{1}{\lambda_3 - \lambda_2}\left[\frac{f(\lambda_3)}{\lambda_3 - \lambda_1}\left(\frac{\partial S}{\partial \lambda_3}\right)^2 + \frac{f(\lambda_2)}{\lambda_2 - \lambda_1}\left(\frac{\partial S}{\partial \lambda_2}\right)^2\right] = 1$$

is separable.

Hence the system is completely integrable!

Remainder: classic Galois theory in 1 slide

Consider a field k and a polynomial $p \in k[x]$. The *splitting field* (SF) $L(p)$ is *the* field extension (modulo isomorphisms) of minimal degree over k in which p splits as $p(x) = \Pi_{i=1}^{\partial p}(x - a_i)$.

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and $\mathbb{Q}[\sqrt[3]{2}, e^{i2\pi/3}]$ is the SF of $p(x) = x^3 - 2$ over $\mathbb{Q}$.

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Consider a field k and a polynomial $p \in k[x]$. The *splitting field* (SF) $L(p)$ is *the* field extension (modulo isomorphisms) of minimal degree over k in which p splits as $p(x) = \Pi_{i=1}^{\partial p}(x - a_i)$.

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E.g. $Aut(L(x^5 - x - 1)/\mathbb{Q}) \simeq S_5$ is not soluble (Artin).

Galois theory of CIS

How to find out whether a Hamiltonian system (M^{2n}, ω, H) is or not a Completely Integrable System?

Galois theory of CIS

How to find out whether a Hamiltonian system (M^{2n}, ω, H) is or not a Completely Integrable System?

The idea is to study the equations of second variations on TM

$$\dot{X}^\alpha(t) = \left.\frac{\partial \xi_H^\alpha}{\partial x^\beta}\right|_{\gamma(t)} X^\beta(t)\,,\ \ X(t) \in T_{\gamma(t)}M$$

defined on integral trajectories γ of the Hamiltonian equations of motion,
where (x^α) are coords on M and (X^α) the variations along γ.

Example: the Hénon-Heiles system

$$H(x,y,p_x,p_y) = \frac{1}{2}\left(p_x^2+p_y^2\right) - y^2(A+x) - \frac{\lambda}{3}x^3$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{p}_x \\ \dot{p}_y \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ y^2+\lambda x^2 \\ 2(A+x)y \end{pmatrix}$$

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Clearly there are orbits with $y(t) = p_y(t) = 0$ for all t.
Along these trjectories the linearized equation writes

$$\begin{pmatrix}\dot{X}\\ \dot{Y}\\ \dot{P}_x\\ \dot{P}_y\end{pmatrix} = \begin{pmatrix}P_x\\ P_y\\ 2\lambda xX\\ 2AY+2xY\end{pmatrix}$$

Theorem (Audin, III.1.12)

If $f : M \to \mathbb{R}$ is a first integral of ξ_H and k is the first order where the k-th order derivative $D^k f : S^k(TM) \to \mathbb{R}$ is not zero on γ, then

$$f^o_\gamma(t, X, P) = D^k f\big|_{\gamma(t)}((X, P), \dots, (X, P))$$

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Indeed, on a solution of the linearized equation, we have that

$$\frac{dH^o_\gamma}{dt} = \dot{p}_x P_x + p_x \dot{P}_x - 2\lambda x \dot{x} X - \lambda x^2 \dot{X} =$$
$$= \lambda x^2 P_x + p_x 2\lambda x X - 2\lambda x p_x X - \lambda x^2 P_x = 0$$

Differential Galois Theory

Definition

Given an algebraically close field k with a derivation D (e.g. $\mathbb{C}(t)$ with d/dt) and a linear ODE

$$\dot{X} = AX\,, \quad A \in M_n(k)\,,$$

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Like in the standard Galois theory, such extension is unique modulo differential isomorphisms.

Definition

The Galois group $Gal(A) \subset GL_n(k)$ of the linear ODE $\dot{X} = AX$ is the group of differential automorphisms of $L(A)$ that fixes k.

Example 1

Consider the linear ODE

$$x' = \frac{\alpha}{t}x$$

on $\mathbb{C}(t)$, namely $A = \left(\frac{\alpha}{t}\right) \in M_1(\mathbb{C}(t))$,
whose solution is $x(t) = t^{\alpha} + c$.

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Then

1. $L(A) \simeq \mathbb{C}(t)$, $Gal(A) \simeq \{1\}$ if $\alpha \in \mathbb{Z}$;
2. $L(A) \simeq \mathbb{C}(t)[u]/\langle u^q - t^p \rangle$, $Gal(A) \simeq \mathbb{Z}_q$ if $\alpha = p/q \in \mathbb{Q}$;
3. $L(A) \simeq \mathbb{C}(t,u)$, $Gal(A) \simeq GL_1(\mathbb{C}) \simeq \mathbb{C}^*$ if $\alpha \notin \mathbb{Q}$.

Example 2 – the Cauchy equation

Consider the Cauchy equation $x'' = \frac{\alpha}{t^2} x$ on $\mathbb{C}(t)$, namely $A = \begin{pmatrix} 0 & 1 \\ \frac{\alpha}{t^2} & 0 \end{pmatrix} \in M_2(\mathbb{C}(t))$, and assume $\alpha \neq -\frac{1}{4}$.

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These two solutions $u_{1,2}$ are independent and if $\sigma \in Gal(A)$ then

$$\sigma(u_i)' = \sigma(u_i') = \sigma\left(\frac{\alpha_i}{t}u_i\right) = \frac{\alpha_i}{t}\sigma(u_i)$$

namely $\sigma(u_i) = \lambda u_i$, $\lambda \in \mathbb{C}$, i.e. all matrices of $Gal(A)$ are diagonal. In particular $Gal(A)$ is abelian.

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Finally consider the Airy equation $x'' = tx$ on $\mathbb{C}(t)$, namely
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Redefine u and v so that their Wronskian is 1. Then, if $\sigma \in Gal(A) \subset GL_2(\mathbb{C})$, a direct calculation shows that, with respecct to the base (u, v),

$$\det \sigma = \det \begin{pmatrix} \sigma(u) & \sigma(v) \\ \sigma(u') & \sigma(v') \end{pmatrix} = \sigma \begin{pmatrix} u & v \\ u' & v' \end{pmatrix} = 1$$

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It can be proved that indeed $Gal(A) \simeq SL_2(\mathbb{C})$.

Three fundamental theorems

Theorem (Morales & Ramis, Audin III.1.13)

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Theorem (Audin III.2.3)

The Galois group of the second variations equation is a symplectic subgroup of $GL(TM)$.

Theorem (Audin III.3.10)

The Lie algebra of the Galois group of the second variation equation of a CIS is abelian.

Example: $H = \frac{1}{2}\left(p_x^2 + p_y^2\right) - y^2(A+x) - \frac{\lambda}{3}x^3$

Apply these results to the Henon-Heiles system.

For $\lambda \neq 0$ we consider the trajectory

$$x(t) = \frac{6}{\lambda t^2}, p_x(t) = \dot{x}(t), y(t) = 0, p_y(t) = 0.$$

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The second variations equations can be reduced to

$$\ddot{X}(t) = 2\left(A + \frac{6}{\lambda t^2}\right)X(t), \lambda \neq 0\ ;\ \ \ddot{X}(t) = tX(t)\,,\ \lambda = 0$$

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The ODE $\ddot{X}(t) = tX(t)$ is the Airy equation. We saw that its Galois group is $SL_2(\mathbb{C})$, so **there cannot be any further integral of motion for** $\lambda = 0$.

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When $A \neq 0$, $\ddot{X}(t) = 2\left(A + \frac{6}{\lambda t^2}\right)X(t)$ is the Whittaker equation. It can be proved that its Galois group is non-abelian when $\frac{6}{\lambda} \neq \frac{k(k+1)}{2}$, $k \in \mathbb{Z}$. This result can be made even stronger:

Theorem (Morales, Thm 6.4)

The Henon-Heiles system is non integrable for $\lambda \neq 1, 2, 6, 16$.

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Note finally that for $A = 0, \lambda = 6$ the HH system is indeed a CIS: $K = 4p_y(xp_y - yp_x) + y^4 + 4x^2y^2$

Related Literature

• J. Moser, E. Zehdner, *Notes on Dynamical Systems*, AMS

• M. Audin, *Hamiltonian systems and their integrability*, AMS, 2008

• J. Morales, *Differential Galois Theory and non-integrability of Hamiltonian systems*, Birkauser, 1983

• V.I. Arnold, A.B. Givental', "Symplectic Geometry", in Dynamical Systems IV: Symplectic Geometry and Its Applications, edited by V.I. Arnold & S.P. Novikov

• M. Dunajski, *Integrable Systems*, Lecture Notes, http://www.damtp.cam.ac.uk/user/md327/ISlecture_notes_2012_part1.pdf

• V. Jurdjevic, *Jacobi's geodesics problem and Lie groups*, Slides, http://www.cmap.polytechnique.fr/geometric-control-srg/slides2/jurdjevic.pdf

• A. Maciejewski and M. Przybylska, *Differential Galois theory and integrability*, Int. J. of Geometric Methods in Modern Physics, 6:8, 2009, arXiv:0912.1046

Hamiltonian Systems close to Integrable

Recall that, if (M^{2n}, ω, H) is a CIS and in a neighborhood $T^n \times D^n \subset T^n \times \mathbb{R}^n$ of a Lagrangian torus invariant by the flow, there exists action-angle coordinates (q, p) , so that $H = H(p)$ and the equations of motion write

$$\dot{q}^\alpha = \frac{\partial H}{\partial p_\alpha}\,,\ \dot{p}_\alpha = 0\,.$$

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If $\dfrac{\partial^2 H}{\partial p_\alpha \partial p_\beta}$ is non-singular at every point, then the n frequencies $\nu(p) = \frac{\partial H}{\partial p_\alpha}(p) : D^n \to \mathbb{R}^n$ label the Lagrangian tori in $T^n \times D^n$.

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Definition

The frequencies $(\nu^1, \dots, \nu^n)$ are non-resonant if there exists $c > 0$ such that

$$|k_\alpha \nu^\alpha| \geq \frac{c}{\|k\|^n}\,,\ \text{ for all } k \in \mathbb{Z}^n \setminus 0.$$

The sets $\Phi_c \subset \mathbb{R}^n$, $c > 0$, of non-resonant frequencies are Cantor sets (closed, perfect and nowhere dense) such that $\mu(\Omega \setminus \Phi_c) = O(c)$ for every bounded $\Omega \subset \mathbb{R}^n$.

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Theorem (Kolmogorov, Arnold, Moser)

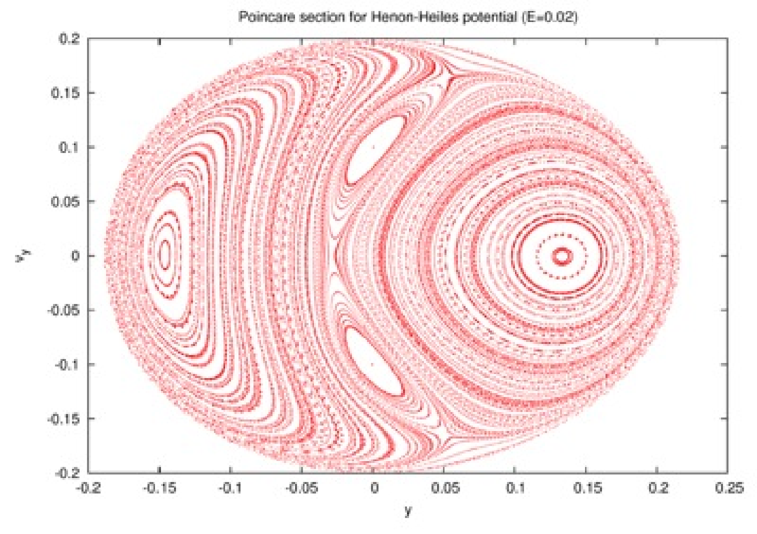
Suppose that (M^{2n}, ω, H) is a CIS, T^n a Lagrangian torus invariant by the flow and a neighborhood where we have Then, if the map $\nu = (\frac{\partial H}{\partial p_\alpha}) : D^n \to \mathbb{R}^n$ is an immersion and the Hamiltonian $H_\varepsilon(q,p) = H(p) + \varepsilon F(q,p)$ is analytic on $T^n \times D^n$, there exists $\delta > 0$ such that for

$$|\varepsilon| < \delta c^2$$

all tori of the unperturbed systems whose frequency ν belongs to Φ_c persists as Lagrangian tori in the perturbed system, being only slightly deformed. Moreover they depend in a Lipschitz way on ν and fill the phase space $T^n \times D^n$ with measure $O(c)$.

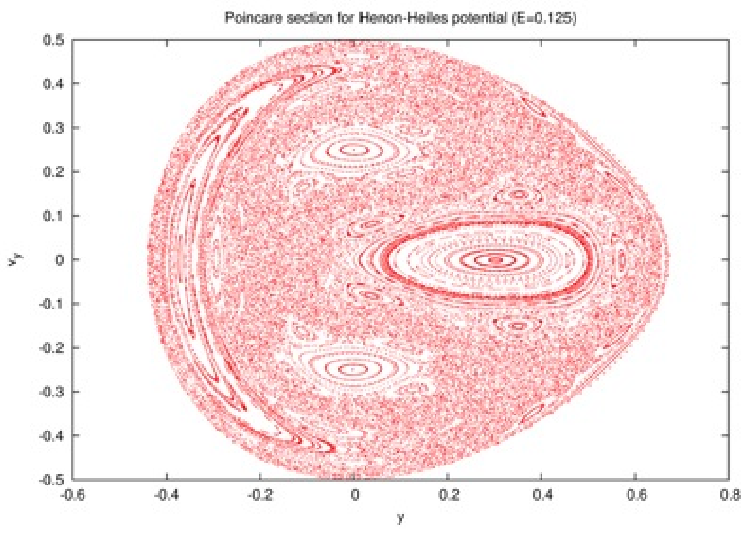
Example: Hénon-Heiles Hamiltonian

Close to integrable...



Example: Hénon-Heiles Hamiltonian

Not so close anymore...



Quantum Hamiltonian Chaos

It was conjectured by Berry and Tabor that the integrability of a Hamiltonian H can be read, in its quantum counterpart $\hat{H}$, from its eigenvalues distribution:

Conjecture (Berry & Tabor)

Let H be a Hamitonian on $\mathbb{R}^n$ and let $P(s)$ the distribution function of the nearest-neighbour spacings $\lambda_{n+1} - \lambda_n$ of the eigenvalues of $\hat{H}$. Then:

1. *if the classical dynamics is* integrable, *then $P(s)$ coincides with the distribution of* uncorrelated *levels with the same mean spacing (Poisson distr.), i.e.*

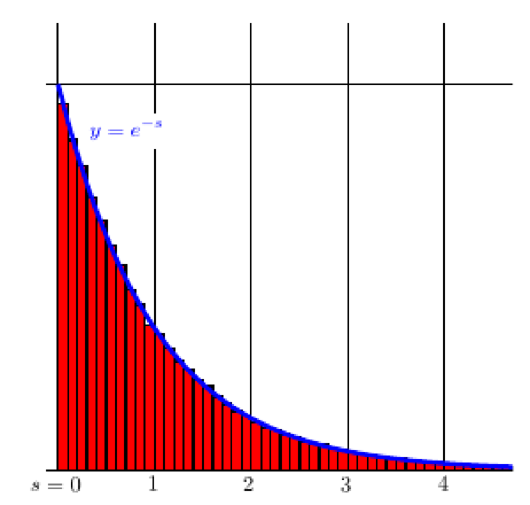
$$P(s) \propto e^{-cs}$$

2. *if the classic dynamics is* chaotic, *then $P(s)$ coincides with the distribution of a suitable ensamble of* random matrices.

Quite interestingly, this conjecture relates Quantum *chaology* with Number Theory, and in particular with the Riemann Zeta

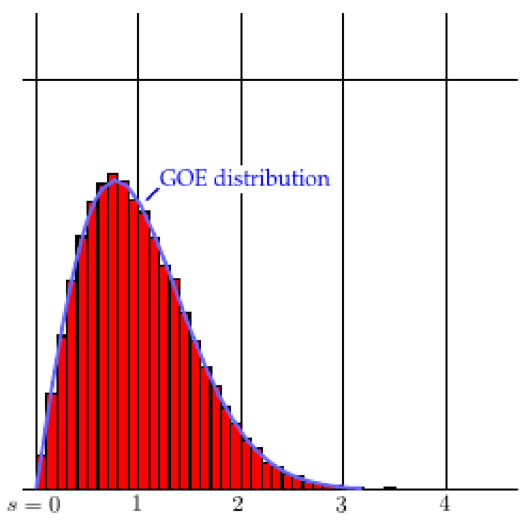
Quantum Hamiltonian Chaos

Poisson distribution:



Quantum Hamiltonian Chaos

GOE distribution:



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Poissonian Systems

Main Definitions and examples

Definition

A Poisson manifold is a pair $(M^n, \{,\})$, where M is a manifold and the bilinear map $\{,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ (Poisson bracket) satisfies the following properties:

1. $\{f,g\} = -\{g,f\}$;
2. $\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0$;
3. $\{f,gh\} = \{f,g\}h + g\{f,h\}$.

Example 1: every symplectic manifold (M^{2n}, ω) is an even-dimensional Poisson manifold with

$$\{f,g\} = \omega(\xi_f, \xi_g)$$

Example 2: on a 3-dimensional Riemannian manifold (M,s), every $h \in C^\infty(M)$ gives rise to the Poisson bracket

$$\{f,g\}_h = \star_s(df \wedge dg \wedge dh)$$

Main Definitions and examples

Like on a symplectic manifold, via the Poisson braket we can associate a vector field ξ_H to each function $H \in C^\infty(M)$ as

$$\xi_H(f) \stackrel{\text{def}}{=} \{H, f\}$$

On a symplectic manifold (M, ω), in a symplectic chart (q^α, p_α),

$$\{q^\alpha, q^\beta\} = 0\,, \quad \{q^\alpha, p_\beta\} = \delta^\alpha_\beta\,, \quad \{p_\alpha, p_\beta\} = 0$$

so that

$$\xi_H(f) = \omega(\xi_H, \xi_f\} = \partial_\alpha H \partial^\alpha f - \partial_\alpha f \partial^\alpha H$$

Clearly ξ_H is the same vector field from the symplectic structure. In case of 3-dim Riemannian manifolds (M, s) and $h \in C^\infty(M)$,

$$\{x^i, x^j\}_h = \sqrt{\det s}\,\varepsilon^{ijk}\,\partial_k h$$

so that

$$\xi_H = \sqrt{\det s}\,\varepsilon^{ijk}\,\partial_j H \partial_k h \partial_i$$

Poisson dynamics

Definition

A *Poissonian system* on the Poissonian manifold $(M, \{,\})$ is given by a smooth function $H \in C^\infty(M)$.

The variation of an observable $f \in C^\infty(M)$ over the flow ϕ^t_H of H is given by

$$\frac{d}{dt} f \circ \phi^t_H = L_{\xi_H} f = \omega(\xi_H, \xi_f) = \{H, f\}$$

This relation is written simply as

$$\dot{f} = \{H, f\}$$

E.g. if the system is symplectic then

$$\dot{q}^\alpha = \{H, q^\alpha\} = \partial^\alpha H\,, \quad \dot{p}_\alpha = \{H, p_\alpha\} = -\partial_\alpha H$$

Integrals of motion

Theorem

f is constant over the integral trajectories of H iff $\{H, f\} = 0$.

In odd dimension $\{,\}$ is degenerate, i.e. there exists observables that commute with all other observables. Such observables are called *Casimirs*.

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$$\dot{x}^i = \{H, x^i\}_h = \sqrt{\det s}\,\varepsilon^{ijk}\partial_j H \partial_k h$$

and h is a Casimir since

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This means that the image of the integral trajectories of H under $\{,\}_h$ are the intersections between the level sets of H and of h.

Example: a Multivalued Poisson DS

Consider $(\mathbb{T}^3, \{,\}_B)$, where $B = B^i(p)dp_i$ is a closed 1-form and

$$\{p_i, p_j\}_B = \varepsilon_{ijk} B^k$$

A direct calculation shows that $\{,\}_B$ is a Poisson structure on $\mathbb{T}^3$.

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The image of the integral trajectories of H are given by the intersections between the level surfaces of H and the leaves of the foliation $B = 0$.

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Their topology instead, i.e. their asymptotics, turns out to be exceptionally rich.

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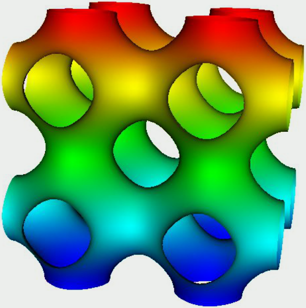
$$H(x,y,z) = \cos x + \cos y + \cos z$$



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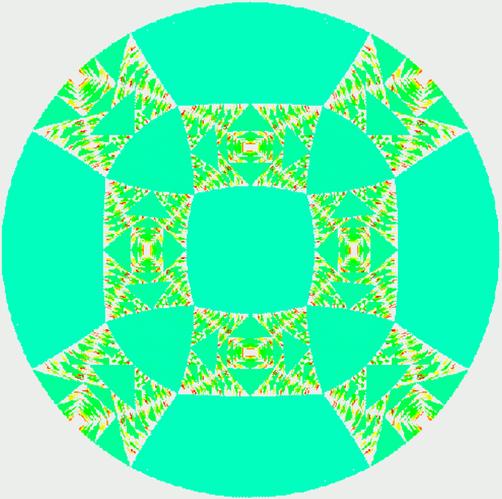

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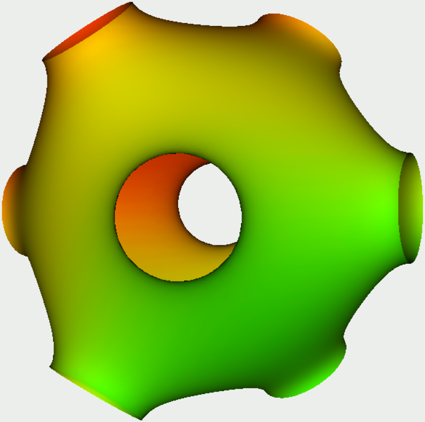

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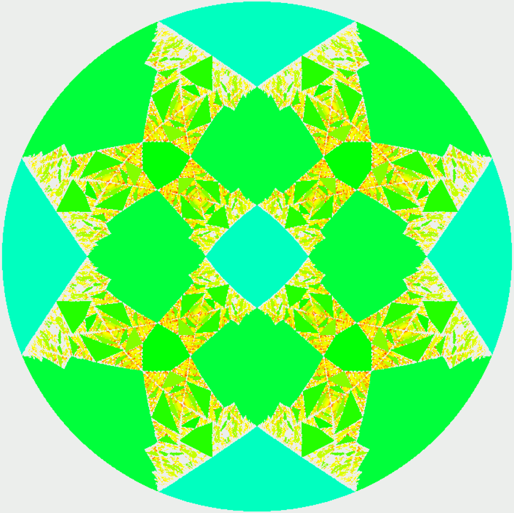

Poisson brackets and QM

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Indeed in QM on $\mathbb{R}^n$ the position and momentum observables $q^\alpha, p_\alpha \in C^\infty(\mathbb{R}^n)$ are replaced resp. by the operators $\hat{q}^\alpha$ (multiplication by q^α) and $\hat{p}_\alpha = \frac{i}{\hbar}\partial_{q^\alpha}$ acting on $L^2(\mathbb{R}^n)$.

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As operators, their commuting relations are

$$[\hat{q}^\alpha, \hat{q}^\beta] = 0\,, \quad [\hat{q}^\alpha, \hat{p}_\beta] = i\hbar\delta^\alpha_\beta\,, \quad [\hat{p}_\alpha, \hat{p}_\beta] = 0.$$

Recall that, in the symplectic setting,

$$\{q^\alpha, q^\beta\} = 0\,, \quad \{q^\alpha, p_\beta\} = \delta^\alpha_\beta\,, \quad \{p_\alpha, p_\beta\} = 0.$$

Poisson brackets and QM

In other words, “$[\hat{f}, \hat{g}] = i\hbar\{f, g\}$”. This analogy is the base of two attempts to fully understanding the relation between CM and QM:

- *geometric quantization* (Souriau, Weinstein, Guillemin, Sternberg...), which uses symplectic geometry to find some natural way to foliate T^*M in Lagrangian leaves (to mimic the separation of q's and p's in QM (polarization);
- *deformation quantization* (Kontsevich, Connes...), which deformes the product in $C^\infty(M)$ in order to get a non-commutative algebra $A_\hbar$ that, in the limit $\hbar \to 0$, reduces to the multiplication in $C^\infty(M)$.

Neither of these attempts, which we have no space to illustrate here, succeeded to date.

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Hamiltonian Systems with Symmetries

Cyclic Coordinates

The form of the Lagrange equations

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In a classical mechanical system in $\mathbb{R}^n$,

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This is the starting point for all results that follow.

First generalization: Noether's Theorem

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Φ induces an action $\hat{\Phi}$ on TM as $\hat{\Phi}(\lambda, q, v) = (q_\lambda, v \cdot \partial_q \Phi(\lambda, q))$

Theorem (Noether, Arnold 20A)

If $L(q, v)$ is invariant by Φ, i.e. if $\hat{\Phi}^* L = L$,

then $p_\Phi(q, \dot{q}) = \xi^\alpha_\Phi(q) \dfrac{\partial L}{\partial \dot{q}^\alpha}(q, \dot{q})$ is a first integral.

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Example: Rotations

Consider the case of $M = \mathbb{R}^3$ and $L(q, v) = \frac{1}{2}\|v\| - V(q)$, with V invariant by rotations, i.e. depending only on the distance of q from the origin
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The action is
$\Phi_z(\lambda, x, y, z) = (x\cos\lambda + y\sin\lambda, -x\sin\lambda + y\cos\lambda)$
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The action is
$\Phi_z(\lambda,x,y,z)=(x\cos\lambda+y\sin\lambda,-x\sin\lambda+y\cos\lambda)$
and $\xi_{\Phi_z}(x,y)=-y\partial_x+x\partial_y$.

The corresponding first integral
(z component of the angular momentum)
is $p_{\Phi_z}(x,y,z,\dot{x},\dot{y},\dot{z})=-y\dot{x}+x\dot{y}$.

Second generalization: Momentum Map

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Every such ξ_a satisfies $\xi_a\{F,G\} = \{\xi_a F, G\} + \{F, \xi_a G\}$
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In both cases, locally ξ_a is the Ham. v.f. of some function H_a.

In order to define the momentum map we must make two assumptions:

1. all ξ_a are Hamiltonian;
2. $\{H_a, H_b\} = H_{[a,b]_{\mathfrak{g}}}$.

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Example: an action on T^*M induced from an action on M is always Poissonian (see Arnold, Appendix 5).

Definition

If Φ is Poissonian, we call *Momentum Map* the map $J : P \to \mathfrak{g}^*$ defined by $J_x(a) = H_a(x)$.

Example: Rotations

Consider the action Φ of SO_3 on $T^*\mathbb{R}^3$ induced by the rotations on the base space.
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This action is Poissonian. The first integrals corresponding to these vector fields are the three components of the angular momentum:

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$$L_x = yp_x - xp_y\,, \quad L_y = zp_y - yp_z\,, \quad L_z = xp_z - zp_x\,.$$

The momentum map is exactly the "angular momentum vector":

$$J(x,y,z,p_x,p_y,p_z) = (L_x, L_y, L_z) \in so(3)^*$$

and $\{L_{x^i}, L_{x^j}\} = \varepsilon_{ijk} L_{x^k} = L_{[x^i,x^j]_{so(3)^*}}$.

Theorem (Covariance of the Momentum Map)

Under J_Φ, the action Φ is taken into the coadjoint action of G on $\mathfrak{g}^*$, namely $J_\Phi(\Phi(g,x)) = Ad^*_{g^{-1}}(J_\Phi(x))$.
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Let g_λ a 1-parameter subgroup of G with Hamiltonian H_a. Then $0 = \frac{d}{d\lambda} H(\Phi(g,x)) = \{H, H_a\}$. □

Symplectic Reduction

Consider a Hamiltonian on a symplectic manifold (P,ω) invariant by some Poissonian action Φ of G on P and set $P_\mu = J_\Phi^{-1}(\mu)$, $\mu \in \mathfrak{g}$.

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Assume Φ satisfies the following properties:

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Theorem (Marsden & Weinstein, Arnold App. 5)

The quotient $M_\mu = P_\mu / G$ is a smooth manifold and inherits from (P,ω) a symplectic structure ω_μ.

Example: Harmonic Oscillator

Consider the action of $\mathbb{S}^1$ on $P = \mathbb{R}^{2n}$ induced by the flow of the Harmonic Oscillator Hamiltonian $H(q,p) = \frac{1}{2}\left(\|p\|^2 + \|q\|^2\right)$.

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All quotient spaces M_μ, $\mu \neq 0$, are symplectomorphic to $\mathbb{C}P^{n-1}$ with a symplectic structure proportional to the Fubini-Study 2-form

$$\omega = \frac{i}{2\pi}\partial\bar{\partial}\ln|z|^2$$

Convexity of the Momentum Map

Theorem (Atiyah, Guillemin, Sternberg (1981))

Consider a Poisson action $\Phi : \mathbb{T}^k \times P^{2n} \to P^{2n}$ on a compact connected symplectic manifold P.
Then $J_\Phi(P) \subset \mathfrak{g}^*$ is a convex polytope.

Example. Consider $P^{2n} = \mathbb{C}P^n$ and $G = \mathbb{T}^{n+1}$ acting on it as $x = (z_1 : \cdots : z_{n+1}) \to (e^{i\theta_1} z_1 : \cdots : e^{i\theta_{n+1}} z_{n+1})$.

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Its image is the simplex

$$\{(s_1, \dots, s_{n+1}) \,|\, s_1 + \dots + s_{n+1} = 1,\ s_1, \dots, s_{n+1} \geq 0\} \subset \mathbb{R}^{n+1},$$

whose vertices are the images of the fixed points $x_i = (0 : \dots : z_i : \dots : 0)$ of the action.

Convexity of multivalued Momentum Maps

Theorem (A. Giacobbe (2000))

Consider a Poisson action $\Phi : \mathbb{T}^k \times P^{2n} \to P^{2n}$ *on a closed connected symplectic manifold* P *with a* multivalued *momentum map* J_Φ*. Then* $J_\Phi(P) \subset \mathfrak{g}^*$ *is a cylinder over a convex polytope.*

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Example. Consider $P^4 = \mathbb{T}^2 \times \mathbb{C}P^1$ with coordinates $((\phi, \psi), (z : w))$ and symplectic structure $\omega = d\phi \wedge d\psi + \frac{i}{2\pi} d\bar{d} \ln |\frac{z}{w}|^2$

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Example. Consider $P^4 = \mathbb{T}^2 \times \mathbb{C}P^1$ with coordinates $((\phi,\psi),(z:w))$ and symplectic structure $\omega = d\phi \wedge d\psi + \frac{i}{2\pi} d\bar{d} \ln |\frac{z}{w}|^2$ and consider the action of $G = \mathbb{T}^3$ on it defined by

$$((\phi,\psi),(z:w)) \to ((\phi+\theta_1,\psi),(e^{i\theta_2}z : e^{i\theta_3}w))$$

The corresponding momentum map is multivalued:

$$J((\phi,\psi),(z:w)) = \left(\psi, \frac{|z|^2}{|z|^2+|w|^2}, \frac{|w|^2}{|z|^2+|w|^2}\right)$$

Its image is $J(\mathbb{T}^2 \times \mathbb{C}P^1) = \mathbb{R} \times S \subset \mathbb{R}^3$,

where $S = \{(s,t) \,|\, s+t = 1,\, s,t \geq 0\}$

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