An example of a fractal set of plane directions having chaotic intersections with a fixed 3-periodic surface

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The problem of the asymptotic behaviour of unbounded connected components of plane sections of a 3-periodic surface in $\mathbb{R}^3$ and the structure of the associated foliations on a surface in the three-torus $\mathbb{T}^3=\mathbb{R}^3/\mathbb{Z}^3$ was posed by S. P. Novikov in [1]. This problem comes from the theory of normal metals and is formulated as follows. In the space $\mathbb{R}^3$ with coordinates (x^1,x^2,x^3) one fixes an embedded surface M that is invariant under shifts by vectors in $\mathbb{Z}^3\subset\mathbb{R}^3$ and has null homologous projection on $\mathbb{T}^3=\mathbb{R}^3/\mathbb{Z}^3$. For all covectors $H=(H_1,H_2,H_3)$ one considers the sections of M by planes $\langle H,x\rangle=\mathrm{const}$ (we call them H-sections) and asks about the asymptotic behaviour of their unbounded regular connected components (if any). This question has been studied in a number of papers [2]–[4]. Physical consequences of the results are discussed in [5].

There are the following three possibilities for the behaviour of the H-sections, and moreover, the type of a section is the same for parallel planes.

Trivial case. All connected components are bounded.

Integrable case. Every regular non-closed component of an H-section is a finitely deformed straight line: $\gamma(t)=t\cdot v+O(1)$, $v\in\mathbb{R}^3\setminus\{0\}$. Moreover, there is a covector $L_{M,H}$, defined up to sign, with relatively prime integral coordinates (L_1,L_2,L_3) that annihilates the vector v. (If we assume our three-space is Euclidean, we can write $v=L_{M,H}\times H$.) The projection of such a component on $\mathbb{T}^3$ is contained in an embedded two-torus whose homology class is Poincaré dual to the covector $L_{M,H}$. We denote the projective class $(L_1:L_2:L_3)\in\mathbb{R}\mathrm{P}^2$ of $L_{M,H}$ by $\ell_{M,H}$.

Chaotic case. The closure of the projection on $\mathbb{T}^3$ of any unbounded component is a surface of genus greater than two. The behaviour of such components has not been studied. In some explicitly described examples such a curve 'wanders all around the plane', that is, the d-neighbourhood of the curve is the whole plane for some finite d. Apparently, such behaviour is typical for the chaotic case, but this has not been proved.

If H_1,H_2,H_3 are linearly dependent over $\mathbb{Q}$, then the definitions above have to be refined. Moreover, in this case the covector $L_{M,H}$ may be not uniquely defined. We skip these details in the present paper.

For a fixed surface M and a rational point $\ell\in\mathbb{Q}\mathrm{P}^2\subset\mathbb{R}\mathrm{P}^2$ let $\mathscr{D}_{M,\ell}$ denote the set $\mathscr{D}_{M,\ell}=\{(H_1:H_2:H_3)\in\mathbb{R}\mathrm{P}^2;\ \ell_{M,H}=\ell\}$.

It is known that for a generic 3-periodic surface M the sets $\mathscr{D}_{M,\ell}$ are disjoint closed domains with piecewise smooth boundary, and the set of directions H with chaotic H-sections has measure zero. (Of course, for some $\ell\in\mathbb{Q}\mathrm{P}^2$ the set $\mathscr{D}_{M,\ell}$ may be empty.) The set of directions H with trivial H-sections is open. We call the non-empty domains $\mathscr{D}_{M,\ell}$ *stability zones.*

For studying the stability zones it is useful to consider a 1-parameter family $M_c=\{x\in\mathbb{R}^3;\ f(x)=c\}$ of level surfaces of a fixed smooth function and to introduce *generalized stability zones* $\mathcal{D}_{f,\ell}=\bigcup_c\mathcal{D}_{M_c,\ell}$. They are also domains with piecewise smooth boundary. If $\ell\neq\ell'$, then the zones $\mathcal{D}_{f,\ell}$ and $\mathcal{D}_{f,\ell'}$ can intersect only at the boundary, and moreover, the set of their common points is at most countable (in all examples known to us, two zones have at most one common point). If there are at least two zones, then there must

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be infinitely many of them. For any point $(H_1 : H_2 : H_3) \in \mathcal{E}(f) = \mathbb{RP}^2 \setminus \bigcup_\ell \mathcal{D}_{f,\ell}$ of the complement there is exactly one level c for which a chaotic regime occurs on M_c, and all the other level surfaces have trivial H-sections. The union $\bigcup_\ell \operatorname{int}(\mathcal{D}_{f,\ell})$ of the interiors of the zones is an open dense subset of $\mathbb{RP}^2$ and its complement $\overline{\mathcal{E}(f)}$ has the form of a 'two-dimensional Cantor set'. However, it is still unknown whether or not $\mathcal{E}(f)$ always has zero measure. According to Novikov's conjecture, the Hausdorff dimension of this set is strictly between 1 and 2.

The facts above are valid for generic smooth and piecewise linear surfaces. A number of examples have been analysed numerically [6]. In this paper we consider one piecewise linear surface and completely describe its stability zones. This surface is called the regular skew polyhedron $\{4,6|4\}$ in [7]. It can be given by the equation $f = 0$ for $f(x^1,x^2,x^3) = \operatorname{mid}\bigl(\cos(2\pi x^1),\cos(2\pi x^2),\cos(2\pi x^3)\bigr)$, where $\operatorname{mid}(a,b,c) = a+b+c-\max(a,b,c)-\min(a,b,c)$. In this case we have $\mathcal{D}_{M_0,\ell} = \mathcal{D}_{f,\ell}$ for all ℓ, and all chaotic H-sections occur at the same level $f = 0$. This is a consequence of the symmetry $f(x^1 + 1/2, x^2 + 1/2, x^3 + 1/2) = -f(x)$ and is not true in the general case. Denote by ψ_1,ψ_2,ψ_3 the following projective transformations: $\psi_1(h_1 : h_2 : h_3) = (h_1 : h_2 + h_1 : h_3 + h_1)$, $\psi_2(h_1 : h_2 : h_3) = (h_1 + h_2 : h_2 : h_3 + h_2)$, $\psi_3(h_1 : h_2 : h_3) = (h_1 + h_3 : h_2 + h_3 : h_3)$.

Proposition 1. *For the surface $M_0 = \{f = 0\}$ the stability zones are*

$$\mathcal{D}_{(1:0:0)}(M_0) = \{(h_1 : h_2 : h_3) \in \mathbb{RP}^2;\ h_1 \geqslant |h_2| + |h_3|\},$$
$$\mathcal{D}_{(1:1:1)}(M_0) = \{(h_1 : h_2 : h_3) \in \mathbb{RP}^2;\ 0 \leqslant h_1 + h_2 + h_3 \leqslant 4h_1, 4h_2, 4h_3\},$$
$$\mathcal{D}_{\psi_{i_1}(\psi_{i_2}(\dots\psi_{i_k}((1:1:1))\dots))}(M_0) = \psi_{i_1}\bigl(\psi_{i_2}(\dots\psi_{i_k}(\mathcal{D}_{(1:1:1)}(M_0))\dots)\bigr),$$

where $(i_1,\dots,i_k)$ is an arbitrary finite sequence of elements in $\{1,2,3\}$, together with all the zones obtained from these by cubic symmetries: permutations and sign changes of coordinates.

Thus, there are three quadrilateral zones $\mathcal{D}_{(1:0:0)}$, $\mathcal{D}_{(0:1:0)}$, $\mathcal{D}_{(0:0:1)}$, and all the rest are triangles. The set $\mathcal{E}(f)$ consists of four parts whose closures are homeomorphic (but not isometric) to the Sierpiński triangle.

Proposition 2. *The intersection $\mathcal{E}(f) \cap \{(h_1 : h_2 : h_3) \in \mathbb{RP}^2;\ h_1,h_2,h_3 > 0\}$ consists of all points of the form $\lim_{k\to\infty} \psi_{i_1}\bigl(\psi_{i_2}(\dots\psi_{i_k}((1:1:1))\dots)\bigr)$, where $(i_1,i_2,\dots)$ runs over all possible sequences of elements in $\{1,2,3\}$ containing each index infinitely many times. The other points in $\mathcal{E}(f)$ are obtained from these by cubic symmetries.*

For example, in the case of the periodic sequence $(1,2,3,1,2,3,\dots)$ we get $(1,\alpha^2-\alpha,\alpha) \in \mathcal{E}(f)$, where α is the real solution of the equation $\alpha^3-\alpha^2-\alpha-1 = 0$.

Proposition 3. *The measure of the set $\mathcal{E}(f)$ is zero.*

We leave the proofs for a more detailed paper. So far we have not been able to estimate the Hausdorff dimension of $\mathcal{E}(f)$ analytically in our example, but several independent numerical computations have given the approximate value ≈ 1.7.

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