Proof of a Dynnikov conjecture on the Novikov problem of plane sections of periodic surfaces

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The problem of Novikov about the asymptotic structure of leaves induced by constant 1-forms in $\mathbb{R}^3$ on 3-ply periodic surfaces has been studied extensively [Zor84, Dyn97, Dyn99, DL03] after the problem was posed in [Nov82].

In particular, fundamental results by A.V. Zorich and I.A. Dynnikov lead to the following picture: the set of directions of 1-forms that induce open leaves on a 3-ply periodic surface M^2 embedded in $\mathbb{R}^3$ is, in the generic case, the disjoint union of a finite number of open subsets $\{\mathcal{D}_l(M^2)\}$ of $\mathbb{RP}^2$, each of whom is labeled by a point $l \in \mathbb{RP}^2$ that is a rational direction with respect to any triple of base symmetry vectors of M^2 (i.e. any set of generators for the group of translations that leaves M^2 invariant). The correspondance between $\mathcal{D}$ and l is given by the following property: the open leaves induced by a 1-form of direction $\omega \in \mathcal{D}_l$ are all strongly asymptotic to a straight line with direction given by the intersection of the straight lines dual of ω and l (i.e. "$\omega \times l$"). All directions falling outside of the $\{\mathcal{D}_l\}$ give rise to closed leaves only (modulo saddle connections).

We call $\{\mathcal{D}_l(M^2)\}$ the set of "stability zones" of M^2. As Dynnikov showed, stability zones corresponding to surfaces that do not intersect each other are compatible, namely zones that overlaps must share the same label, so that it is also possible to associate a set of "stability zones" $\{\mathcal{D}_l(f)\}$ to any Morse 3-ply periodic function f, defining it as the union of the stability zones of all of its level sets. In this case, the set of "stability zones" either is a single set covering the whole projective plane or contains an infinite countable number of elements distributed, loosely speaking, in a fractal-like manner.

In the second case $\bigcup \mathcal{D}_l(f)$ is dense in $\mathbb{RP}^2$ but it does not cover it fully: its complement is the union of all zones boundaries plus a set of "ergodic" directions; these last directions form a closed perfect set with empty interior (whose measure is still unknown) and induce leaves with complicated topology at a single level of f and only closed leaves at every other level. Recently [DL03b] we showed that, when we are in the second case, the limit points of the set of all labels are exactly the set of all boundary points and all ergodic directions. Our proof relies on a Dynnikov's claim based on non-rigorous arguments; the goal of this communication is to provide a full proof for it.

Let us point out first of all that, once a symmetry base for M^2 is chosen, every label l can be represented uniquely (modulo sign) by a triple L of indivisible integers. This correspodance provides us a "norm" for such points (as the norm of the corresponding vector L). Now, be $d(l, l')$ the distance on $\mathbb{RP}^2$ given by the smallest angle between the straight lines l and l':

Lemma 1. *For any 3-ply periodic surface M there exists a stricly-positive finite constant C s.t. $d(\mathcal{D}_l(M), l) \leq C/\|l\|$.*

Proof. Be ω a constant 1-form with a generic direction $l' \in \mathcal{D}_l(M^2)$; then [Dyn97] all open leaves induced on M^2 by ω lie on periodic submanifolds with boundary $\bar{N}_i(l')$ that are periodic with respect to the rank-2 sublattice $\ker l$ and whose boundary is a set of disjoint isolated flat discs (the critical leaves separating open orbits from closed ones). By filling the holes in the natural way we get genus-zero manifolds N_i that are a finite periodic deformation of a plane of direction l.

We claim that $d(l, l') \leq C/\|l\|$ for some constant finite positive constant C depending only on M. Indeed, be N a fundamental domain for any of the N_i and let us project it orthogonally over the plane π of direction $l' \times (l \times l')$. On one side, since π is perpendicular to the flat discs on N, we get an upper bound for the area of the projection as $\mathcal{A}(N_\pi) = \mathcal{A}((N \setminus \{Discs\})_\pi) \leq \mathcal{A}(N \setminus \{Discs\}) \leq \mathcal{A}(M^2) := C/2 < \infty$. On the other side, since by repeating N periodically we get a surface that is a finite deformation of a plane of direction l (whose fundamental domain has area $\|l\|$), then also $\mathcal{A}(N_\pi) \geq \|l\| \sin d(l, l')$, as any finite deformation of a plane can only increase its projection in any direction. These two facts imply that $\sin d(l, l') \leq C/2\|l\|, \forall l' \in \mathcal{D}_l(M^2)$, and therefore $d(l, l') \leq C/\|l\|$ for l big enough. ∎

Theorem 1. *For every 3-ply periodic Morse function f there is a finite constant C_f such that every stability zone $\mathcal{D}_l(f)$ is contained inside a circle of radius $C_f/\|l\|$ centered at l.*

Proof. Using Lemma 1 we see that, for every level $M_e = f^{-1}(e)$, the set $\mathcal{D}_l(M_e^2)$ is contained in a circle of radius $C_e = 2\mathcal{A}(M_e^2)$ centered at l, so $\mathcal{D}_l(f) = \cup_{e\in\mathbb{R}}\mathcal{D}_l(M_e^2)$ is contained inside a disc of radius $C_f = \sup_{e\in\mathbb{R}} C_e$, that is finite because C_e is a continous function of e with compact support. $\square$

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