On the exponential growth of norms in semigroups of linear endomorphisms and the Hausdorff dimension of attractors of projective Iterated Function Systems.

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Abstract

Given a free finitely generated semigroup S of the (normed) set of linear maps of a real or complex vector space V into itself, we provide sufficient conditions for the exponential growth of the number $N(k)$ of elements of S contained in the sphere of radius k as $k \to \infty$ and we relate the growth rate $\lim_{k\to\infty} \log N(k)/\log k$ to the exponent of a zeta function naturally defined on S. When $V = \mathbb{R}^2$ (resp. $\mathbb{C}^2$) and S is a semigroup of volume-preserving maps, we also relate this growth rate to the Hausdorff dimension of the attractor of the induced semigroup of automorphisms of $\mathbb{R}P^1$ (resp. $\mathbb{C}P^1$).

1 Introduction

The basic problem of the asymptotic behaviour of the norms of products of some fixed finite set of square matrices has been extensively studied in the context of the theory of random matrices. In particular, in a celebrated paper [FK60], Furstenberg and Kesten proved that, given some finite number of square matrices A_i and under some suitable measure, the norm of almost

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all products of k of the A_i grows as γ^k, where γ is some Lyapunov exponent associated to the A_i.

In this paper, we address the same subject from a different point of view. We consider a choice of finitely many (possibly repeated) real or complex matrices A_i such that there are only finitely many of their products whose norm is not larger than k for every $k > 0$ and denote this number by $N(k)$. We are interested in the asymptotics of $N(k)$ for $k \to \infty$. Equivalently, we are interested in the asymptotics of the sequence of norms of the products of the A_i when they are sorted in monotonically non-decreasing order. As an application, we show how, in particular cases, these asymptotics are related to the theory of real and complex self-projective Iterated Function Systems, whose rich geometrical and topological properties have been recently investigated in the real case by M.F. Barnsley and A. Vince [BV12] and in the complex case by A. Vince [Vin13].

A simple example, containing already the full complexity of the problem, is given by the free semigroup $\boldsymbol{C} \subset SL_2(\mathbb{R}^+)$ generated by

$$C_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \, C_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

While the generators of $\boldsymbol{C}$ are *parabolic* elements of $SL_2(\mathbb{R}^+)$, so that the norm of their powers grows linearly, their products

$$C_2 C_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \, C_1 C_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

are *hyperbolic*, and therefore the norms of their powers grow exponentially. The log-log plot of the norms of elements $C_n \in \boldsymbol{C}$ in lexicographic order shows clearly this difference in speed growths (see Fig. 1, left): the fastest products grow exponentially with rate asymptotically equal to the *joint spectral radius* [Jun09], which in this case equals the Golden Ratio g, while the slowest grow only linearly. On the other side, the log-log plot of the norms ordered monotonically with the norm suggests a fast convergence to a straight line (see Fig. 1, right). Since this can be seen, loosely speaking, as the log-log plot of the inverse of $N(k)$, it suggests that the limit of $\log N(k)/\log k$ for $k \to \infty$ exists and is bounded.

To the best of our knowledge this problem was first studied (implicitly) by D.W. Boyd in a series of papers dedicated to the determination of the Hausdorff dimension of the Apollonian gasket [Boy70, Boy71, Boy72, Boy73a,

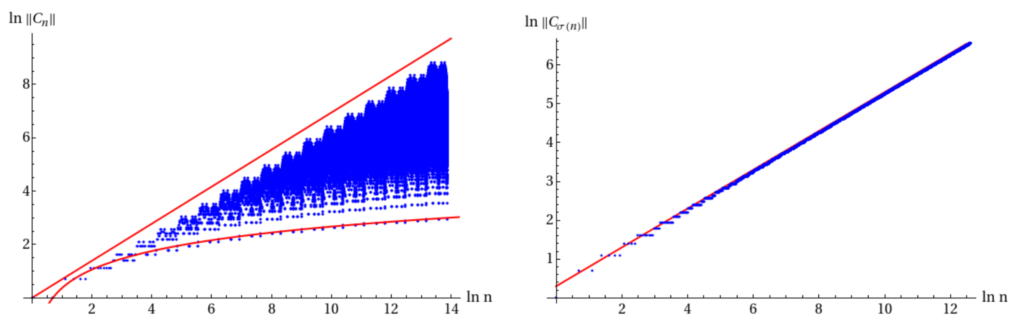

Figure 1: (Left) Log-log plot of the norms of the first 1048574 elements of $\boldsymbol{C}$ in *lexicographic* order. The norms (in blue) are asymptotically bounded by the curves $y = gx$, where g is the Golden Ratio, and $y = \ln x - \ln \ln 2$ (in red). (Right) Log-log plot of the norms of the first 300000 elements of $\boldsymbol{C}$ reordered, according to some permutation $\sigma(n)$, in *non-decreasing* norm order (in blue). The norms appear to tend asymptotically to the line $y = 0.5x + 0.3$ (in red).

Boy73b, Boy82]. Indeed, the geometry of the Apollonian gasket can be described through the norms of the products of three unimodular 4×4 matrices H_i (introduced by K.E. Hirst in [Hir67]) which freely generate a semigroup $\boldsymbol{H}$. Boyd proved that the quantity $d = \lim_{k\to\infty} \log N(k)/\log k$ (where, as above, $N(k)$ is the number of elements of $\boldsymbol{H}$ with norm not larger than k) exists, is finite and coincides with the Hausdorff dimension of the gasket. Recently Kontorovich and Oh strenghtened this result by showing that, for the semigroup $\boldsymbol{H}$, $N(k) \asymp k^d$, i.e. there are constants $A, B > 0$ such that $Ak^d \leq N(k) \leq Bk^d$ for all k [KO11].

In Section 2, we generalize Boyd's ideas and techniques and provide a general sufficient condition for the existence and boundedness of the limit above for a free semigroup of (not necessarily invertible) square matrices (Theorem 3). As a byproduct, we define on semigroups of square matrices a natural zeta function whose critical exponent does not depend on the particular norm chosen (Theorem 1) and equals the limit above (Theorem 3). We provide an alternate characterization of this exponent in terms of partial sums of the zeta function restricted to the elements of the semigroup which are products of some fixed number of the semigroup generators (Theorem 2). We leave to a future paper the problem of finding sufficient condition for $N(k)$ to be asymptotic to some power of k as in case of the Apollonian gasket.

In Section 3, in the spirit of a seminal paper of Sullivan [Sul84] which relates the Hausdorff dimension of the attractor of a geometrically finite

Kleinian group to a critical exponent defined in the context of hyperbolic geometry, we consider the action induced by a free semigroup of unimodular 2×2 real (resp. complex) matrices on $\mathbb{R}P^1 \simeq \mathbb{S}^1$ (resp. $\mathbb{C}P^1 \simeq \mathbb{S}^2$) and show that, under suitable natural conditions, the Hausdorff dimension of the attractor of this action is determined by the critical exponent of the semigroup (Theorem 4). We leave to a future paper the study of the case of $n \times n$ matrices for $n \geq 3$.

2 Asymptotic growth of norms

We endow the vector space $M_n(K)$, $K = \mathbb{R}$ or $\mathbb{C}$, of all $n \times n$ matrices with coefficients in K with the max norm, i.e., given a matrix $M = (M^i_j)$, we set $\|M\| = \max_{i,j=1,\dots,n} \left\{ \left| M^i_j \right| \right\}$. Note that this norm is not sub-multiplicative:

$$\sup_{P,Q \in M_n(K)} \frac{\|PQ\|}{\|P\|\|Q\|} = n\,. \tag{1}$$

Since in finite dimension all norms are equivalent, the main results of the paper will not depend on this particular choice.

2.1 The multi-indices semigroup.

We denote by $\mathcal{I}^m$ the infinite m-ary tree of multi-indices of integers ranging from 1 to m defined as follow. The root of the tree is the number 0. The m children of 0 (the *1-indices*, whose set we denote by $\mathcal{I}^m_1$) are the integers from 1 to m. Their children (the *2-indices*, whose set we denote by $\mathcal{I}^m_2$) are the ordered pairs $1i, \dots, mi$, $i \in \mathcal{I}^m_1$, and so on recursively for the *k-indices*, $k > 2$, which we denote by $\mathcal{I}^m_k$. Since we will use them often, we denote by $\mathcal{D}^m_\ell$, $\ell \geq 0$, the sets of all *diagonal* multi-indices $I = i_1 \dots i_\ell \in \mathcal{I}^m$, i.e. such that $i_1 = \cdots = i_\ell$, and set $\mathcal{D}^m = \cup_{\ell \geq 0} \mathcal{D}^m_\ell$. Similarly, we denote by $\mathcal{J}^m_\ell$, $\ell \geq 2$, the set of all *next-to-diagonal* multi-indices $I = i_1 i_2 \dots i_\ell \in \mathcal{I}^m$, i.e. those such that $i_1 \neq i_2 = \cdots = i_\ell$, and set $\mathcal{J}^m = \cup_{\ell \geq 2} \mathcal{J}^m_\ell$.

We endow $\mathcal{I}^m$ with the canonical structure of the semigroup given by $i_1 \dots i_k \cdot i'_1 \dots i'_{k'} = i_1 \dots i_k i'_1 \dots i'_{k'}$ with 0 as identity element. Finally, we denote by $I' = i_1 \dots i_k$ the k-index obtained from the $(k+1)$-index $I = i_0 i_1 \dots i_k$ by dropping the first index on the left.

2.2 Gaskets of matrices.

Given m matrices $A_1, \dots, A_m \in M_n(K)$, we denote by $\boldsymbol{A} = \langle A_1, \dots, A_m \rangle$ the semigroup they generate. In this paper, we are mainly interested in the asymptotic growth of norms of matrices in free semigroups but, since our results hold for the more general case when there are relations between the generators *as long as we take into account the multiplicity of each element*, we formulate most theorems of the present paper in terms of semigroup homomorphisms $\mathcal{I}^m \to M_n(K)$. We often denote such objects with the letter $\mathcal{A}$ and use the notation

$$A_I \stackrel{\text{def}}{=} \mathcal{A}(i_1 \dots i_k) = A_{i_1} \cdot \dots \cdot A_{i_k},$$

where $A_i \stackrel{\text{def}}{=} \mathcal{A}(i)$, $i = 0, 1, \dots, m$. We say that the matrices $A_1, \dots, A_m$ *generate* $\mathcal{A}$.

Note that A_0 can be any idempotent matrix such that $A_0 A_i = A_i A_0 = A_i$ for every $i = 0, \dots, m$; in this paper A_0 will always be the identity matrix $\mathbb{1}_n$. The space of all semigroup homomorphisms with $A_0 = \mathbb{1}_n$ can be naturally identified with $(M_n(K))^m$ and inherits from this space the structure of vector normed space. We shall use the following $\|\mathcal{A}\| = \max_{1 \leq i \leq m}\{\|A_i\|\}$.

When there is no relation between the A_i, there is a bijection between $\mathcal{A}(\mathcal{I}^m)$, the image of $\mathcal{I}^m$ in $M_n(K)$ via $\mathcal{A}$, and $\{\mathbb{1}_n\} \cup \boldsymbol{A}$. In other words, when the A_i are free generators, it is essentially equivalent referring to either the semigroup homomorphism $\mathcal{A}$ or the semigroup $\boldsymbol{A}$. Given any $M \in GL_n(K)$, we denote by $\mathcal{A}_M$ the "right coset" map defined by $\mathcal{A}_M(I) \stackrel{\text{def}}{=} A_I M$. If $\boldsymbol{A}$ is free, then there is a bijection between $\mathcal{A}_M(\mathcal{I}^m)$ and $\{M\} \cup \boldsymbol{A}M$, where $\boldsymbol{A}M$ is a right coset of $\boldsymbol{A}$. Clearly $\mathcal{A}_{\mathbb{1}_n} = \mathcal{A}$.

Definition 1. *Given a semigroup homomorphism $\mathcal{A} : \mathcal{I}^m \to GL_n(K)$, we denote by $N_{\mathcal{A}_M}(r)$ the cardinality of the set of matrices in $\mathcal{A}_M(\mathcal{I}^m)$ whose norms are not larger than r and say that $\mathcal{A}_M$ is a m-*gasket *(or simply a* gasket*) if $N_{\mathcal{A}_M}(r) < \infty$ for every $r > 0$. Moreover, we say that the gasket $\mathcal{A}_M$ is* hyperbolic *if the sequence $a_k = \min_{I \in \mathcal{I}^m_k} \|A_I M\|$ diverges exponentially, namely if there exists $\alpha > 1$ such that $a_k \asymp \alpha^k$, where $\asymp$ means that the ratio of the terms on either side is bounded away from 0 and ∞ for all k. When a_k is slower than exponential we say that $\mathcal{A}_M$ is* parabolic[1].

The proofs of the next four statements are immediate and we omit them.

[1]Note that a_k cannot be faster than exponential so this covers all possible cases.

Proposition 1. *If $\mathcal{A}_M$ is a gasket with respect to some norm, it is a gasket with respect to every norm. Similarly, $\mathcal{A}_M$ is a gasket iff $\mathcal{A}$ is.*

Proposition 2. *A necessary condition for $\mathcal{A}_M$ to be a gasket is that each of its generators A_i have either an eigenvalue of modulus larger than 1 or a non-trivial Jordan block with respect to an eigenvalue of modulus 1.*

Corollary 1. *For every $\mathcal{A}$, we can find a small enough $\lambda > 0$ such that the homomorphism $\lambda\mathcal{A}$, generated by the matrices λA_i, is not a gasket.*

Lemma 1. *Let $\mathcal{A}$ be a gasket. Then $\lambda\mathcal{A}$ is also a gasket for every $\lambda \geq 1$.*

For every 1-dimensional subspace $\mathcal{V} \subset (M_n(K))^m$, we call *critical radius* the number $\rho_{\mathcal{V}} = \sup_{\mathcal{A}\in\mathcal{V}}\{\|\mathcal{A}\|, \mathcal{A} \text{ is not a gasket}\}$. By Corollary 1, every direction has a critical radius larger than 0 (possibly infinite). By Proposition 2, all $\mathcal{A} \in \mathcal{V}$ with $\|\mathcal{A}\| < \rho_{\mathcal{V}}$ are not a gasket. Nothing can be said in general when $\|\mathcal{A}\| = \rho_{\mathcal{V}}$ (see Example 4).

We write sometimes $\rho_{\mathcal{A}}$ to indicate the critical radius of the 1-dimensional subspace generated by $\mathcal{A} \neq 0$. Next example shows that, if $\mathcal{A}$ is generated by invertible matrices, the corresponding critical radius is always finite.

Example 1. *Every homomorphism $\mathcal{A} : \mathcal{I}^m \to GL_n(\mathbb{C})$ whose generators A_i have all their spectrum outside the open unit disc and their largest modulus eigenvalue λ_i outside the unit circle is a hyperbolic gasket. Indeed no A_i shortens vectors and so, for every $k \in \mathbb{N}$, $\|A_I\| \geq \min_{1\leq i\leq n} |\lambda_i|^k$ when $|I| \geq km$.*

Example 2. *This elementary but still non-trivial example was suggested to the author by I.A. Dynnikov and was the starting point for the author's study of the asymptotics of norm's growth in semigroups of linear maps in full generality. The (free) semigroup $\boldsymbol{C} \subset SL_2(\mathbb{N})$ generated by the two* parabolic *(i.e. with trace equal to ± 2) matrices C_1 and C_2 mentioned in the introduction is a parabolic gasket. It is a gasket because, if $M \in SL_2(\mathbb{N})$ is distinct from the identity, then $\|C_i M\| \geq \|M\| + 1$ since M has at least a column with both entries different from 0. It is parabolic because*

$$\min_{I\in\mathcal{I}^m_k}\{\|C_I\|\} \leq \|C_1^k\| = k.$$

Note that, as we already pointed out in the introduction, $\boldsymbol{C}$ contains also hyperbolic *elements, so that in the sets $\boldsymbol{C}_k = \{C_I, |I| = k\}$ there are some elements whose norm grows polynomially and some others whose norm grows exponentially with k. Note, finally, that $\|\boldsymbol{C}\| = 1$ and $\rho_{\boldsymbol{C}} = 1$.*

Example 3. *Suppose that $A_1, \dots, A_m \in M_n(Z)$, where $Z = \mathbb{Z}$ or $\mathbb{Z}[i]$, generate* freely $\boldsymbol{A}$. *Then $\boldsymbol{A}$ is a gasket, since in $M_n(Z)$ there are only finitely many matrices whose norm is smaller than any fixed $r > 0$ and, by the freedom hypothesis, the products of any number of A_i are all distinct.*

Definition 2. *Let $\mathcal{A} : \mathcal{I}^m \to GL_n(K)$ be a gasket and $M \in GL_n(K)$. We call* zeta function *of $\mathcal{A}_M$ the series*

$$\zeta_{\mathcal{A}_M}(s) = \sum_{I \in \mathcal{I}^m} \|A_I M\|^{-s}.$$

We call exponent *of $\mathcal{A}_M$ the number $s_{\mathcal{A}_M}$ defined as follows:*

$$s_{\mathcal{A}_M} = \sup_{s \geq 0} \{ s \,|\, \zeta_{\mathcal{A}_M}(s) = \infty \}.$$

Note that, if $s_{\mathcal{A}_M} < \infty$, we also have that $s_{\mathcal{A}_M} = \inf_{s \geq 0} \{ s \,|\, \zeta_{\mathcal{A}_M}(s) < \infty \}$.

The elementary inequality $n\|P\|\|M\| \geq \|PM\| \geq \|P\|/(n\|M^{-1}\|)$ grants the following:

Proposition 3. *Let $\mathcal{A} : \mathcal{I}^m \to M_n(K)$ be a gasket. The exponent $s_{\mathcal{A}}$ does not depend on the particular norm used in $M_n(K)$. Similarly, $s_{\mathcal{A}_M} = s_{\mathcal{A}}$ for all $M \in GL_n(K)$.*

Note that the series $\zeta_{\mathcal{A}_M}(s)$ surely diverges for all s if $\mathcal{A}_M$ is not a gasket. On the other side, next example shows that the property of being a gasket is not sufficient for its convergence.

Example 4. *Let $\mathcal{A}$ be the (parabolic) gasket generated by*

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ A_2 = A_1 \in SL_2(\mathbb{N}).$$

In this case $\|A_I\| = |I|$, so that $\zeta_{\mathcal{A}}(s) = \sum_{k \geq 0} 2^k k^{-s}$ diverges for all s, i.e. $s_{\mathcal{A}} = \infty$. Note that $\|\mathcal{A}\| = 1$ and $\rho_{\mathcal{A}} = 1$, namely in the subspace generated by $\mathcal{A}$ elements with norms equal to the critical radius are gaskets.

Now, let $\mathcal{B}$ be the (hyperbolic) gasket generated by

$$B_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},\ B_2 = B_1 \in SL_2(\mathbb{N}).$$

Then $\|B_I\| = F_{2|I|+2}$, where $F = (0, 1, 1, 2, 3, 5, \dots)$ is the Fibonacci sequence. Hence, asymptotically, $\|B_I\| \simeq g^{2|I|}$, where $g = \frac{1+\sqrt{5}}{2}$ is the golden *ratio, and so $\zeta_{\mathcal{B}}(s)$ diverges or converges with $\sum_{k\geq 0} 2^k g^{-2sk}$, i.e. $s_{\mathcal{B}} = \frac{1}{2\log_2 g}$. Note, finally, that in this case we have $\|\mathcal{B}\| = 2$ and $\rho_{\mathcal{B}} = 2/g^2$. Since both generators of the homomorphism $\rho_{\mathcal{B}}\mathcal{B}/2$ have 1 as their largest eigenvalue, in this case the elements of the subspace generated by $\mathcal{B}$ with norm equal to the critical radius are not gaskets.*

Showing that all hyperbolic gaskets have a finite exponent does not require any effort:

Proposition 4. *Let $\mathcal{A} : \mathcal{I}^m \to M_n(K)$ be a hyperbolic gasket. Then $s_{\mathcal{A}}$ is finite.*

Proof. Since $\mathcal{A}$ is hyperbolic, then $\|A_I\| \geq A\alpha^{|I|}$ for some $A > 0$ and $\alpha > 1$. Hence

$$\zeta_{\mathcal{A}}(s) \leq \sum_{k=0}^{\infty} m^k A^{-s}\alpha^{-ks} = \sum_{k=0}^{\infty} A^{-s} m^{k(1-s\log_m \alpha)},$$

so that $\zeta_{\mathcal{A}}(s) \leq \infty$ for $s \geq \log_\alpha m$, namely $s_{\mathcal{A}} \leq \log_\alpha m$. $\square$

As Example 4 shows, in order to prove a similar statement for the parabolic case, we must require some growth condition on the norms of products.

Definition 3. *We say that a homomorphism $\mathcal{A}_M : \mathcal{I}^m \to M_n(K)$, $M \in GL_n(K)$, is* fast *if there is a constant $c > 0$ such that*

$$\|A_{IJK}M\| \geq c\|A_I M\|\|A_{JK}M\| \tag{2}$$

for every multi-index $I, K \in \mathcal{I}^m$ and $J \in \mathcal{J}^m$. We call

$$c_{\mathcal{A}_M} = \inf_{\substack{I,K\in\mathcal{I}^m \\ J\in\mathcal{J}^m}} \frac{\|A_{IJK}M\|}{\|A_I M\|\|A_{JK}M\|}$$

the coefficient *of the homomorphism.*

Note that inequality (2) is homogeneous and therefore, if it is valid for a homomorphism, it is valid for every homomorphism proportional to it. Notice, moreover, that the coefficient of a gasket depends on the particular norm chosen but, because of the topological equivalence of norms in finite dimension, the property of being fast does not:

Proposition 5. *Let $M \in GL_n(K)$. Then, if $\mathcal{A}_M$ is fast for some norm, it is fast for all norms. Similarly, $\mathcal{A}_M$ is fast iff $\mathcal{A}$ is.*

Example 5. *Consider the parabolic and hyperbolic gaskets of Example 4. Since*

$$\|A_1^{k'} A_2 A_1^k\| = \|A_1^{k+k'+1}\| = 1 + k + k'$$

and $\inf_{k,k' \geq 1}\{(1+k+k')/(kk')\} = 0$, *it follows that* $\mathcal{A}$ *is not fast.*

On the contrary, since any product of $N = k + k'$ *copies of* $B_{1,2}$ *is equal to* B_1^N *and*

$$\|B_1^{k'} B_1^k\| = \|B_1^{k+k'}\| = F_{2k+2k'} > F_{2k'} F_{2k-2} = \|B_1^{k'}\|\|B_1^{k-1}\|,$$

then $\mathcal{B}$ *is fast with* $c_{\mathcal{B}} \geq 1$. *To see that in fact* $c_{\mathcal{B}} = 1$ *it is enough to consider the case* $k' = 1$, $k \to \infty$.

Example 6. *The (parabolic) cubic gasket* $\boldsymbol{C}$ *of Example 2 is fast. Indeed, consider first* $J = 21L$, *with* $C_L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, *so that*

$$C_J = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ a+2c & b+2d \end{pmatrix}.$$

Clearly, $\|C_J\| = \max\{a+2c, b+2d\} \leq 2\max\{a+c, b+d\}$ *and, therefore,*

$$\|MC_J\| \geq \frac{1}{2}\|M\|\|C_J\|$$

for every $M \in SL_2(\mathbb{N})$. *The same argument applies to* $J = 12L$. *Since* $\|C_1^{k'} C_1 C_2 C_1^k\| = k'(k+1)$, $\|C_1^{k'}\| = k'$ *and* $\|C_1 C_2 C_1^k\| = 2k+1$, *it follows at once that in fact* $c_{\boldsymbol{C}} = 1/2$.

Example 7. *A hyperbolic gasket is not necessarily fast. Consider for instance* $\mathcal{A} : \mathcal{I}^2 \to GL_2(\mathbb{Z}[i])$ *with*

$$A_1 = \begin{pmatrix} 2 & 0 \\ 0 & \alpha \end{pmatrix}, A_2 = \begin{pmatrix} \beta & 0 \\ 0 & 2 \end{pmatrix},$$

where $|\alpha|, |\beta| = 1$. $\mathcal{A}$ *is hyperbolic since every* A_I, $|I| = 2k$, *contains an entry with modulus* $2^{k'}$ *and* $k' \geq k$. *On the other side,* $\mathcal{A}$ *is not fast. Indeed,* $\|A_1^k A_1 A_2^{k'}\| = 2^{-k}\|A_1^k\|\|A_1 A_2^{k'}\|$ *for all* $k \leq k'$ *and so*

$$\inf_{I \in \mathcal{I}^2, J \in \mathcal{J}^2} \frac{\|A_{IJ}\|}{\|A_I\|\|A_J\|} = 0.$$

2.3 Exponent of a fast gasket. The case $m = 1$.

Let us briefly discuss the exponent of a gasket in the trivial case $m = 1$, since these results will be useful later in this section.

Here $\mathcal{I} = \mathbb{N}$, $\mathcal{A}$ is generated by a single matrix $A \in M_n(K)$, $K = \mathbb{R}$ or $\mathbb{C}$, and $\mathcal{A}(k) = A^k$. In order for $\mathcal{A}$ to be a gasket, then, it is necessary and sufficient that either A has an eigenvalue with modulus larger than 1 (in which case it will be a hyperbolic gasket) or that it has a non-trivial Jordan block with respect to an eigenvalue of modulus 1 (in which case it will be a parabolic gasket).

If $\mathcal{A}$ is hyperbolic, then $\|A^k\|$ grows exponentially with k and so $s = 0$ is the only exponent that can make the series $\zeta_{\mathcal{A}}(s) = \sum_{k\in\mathbb{N}} \|A^k\|^{-s}$ divergent, i.e. $s_{\mathcal{A}} = 0$.

If $\mathcal{A}$ is parabolic, then its generator A has no eigenvalue of modulus larger than 1 and a Jordan block of maximal size $d + 1$ with eigenvalue of modulus 1, so that its norm grows as some polynomial of degree $d < n$. Hence, $\zeta_{\mathcal{A}}(s)$ diverges for $s \leq 1/d$ and is finite for $s > 1/d$, i.e. $s_{\mathcal{A}} = 1/d$.

Now, consider the number $N_{\mathcal{A}}(r)$ of powers of A whose norm is not larger than r. When $\|A^k\|$ is a polynomial of order d, their number grows as $r^{1/d}$, so that the limit $\lim_{k\to\infty} \log N_{\mathcal{A}}(r)/\log r = 1/d$ equals $s_{\mathcal{A}}$. When $\|A^k\|$ grows exponentially, then $N_{\mathcal{A}}(r)$ grows logarithmically and therefore $\lim_{k\to\infty} \log N_{\mathcal{A}}(r)/\log r = 0$ again equals $s_{\mathcal{A}}$. Theorems 1 and 3 shall extend these results to the case $m > 1$ under suitable conditions.

2.4 Exponent of a fast gasket. The case $m > 1$.

When there is more than one generator, things are qualitatively different: the number of terms of order k (i.e. products of k generators) increases exponentially and there can be coexistence of polynomial and exponential growths of norms for terms of the same order (e.g. see Example 6).

Generalizing Boyd's arguments in [Boy73b], we will show in this section how to build upper and lower bounds for $\zeta_{\mathcal{A}_M}(s)$ in terms of the series of the norms of the "diagonal" terms $A_D M$, $D \in \mathcal{D}^m$, and of those "next-to-diagonal" ones $A_J M$, $J \in \mathcal{J}^m$. The key point of the next arguments is the following elementary recursive re-writing of the zeta function:

$$\zeta_{\mathcal{A}_M}(s) = \sum_{D\in\mathcal{D}^m} \|A_D M\|^{-s} + \sum_{J\in\mathcal{J}^m} \zeta_{\mathcal{A}_{A_J M}}(s). \tag{3}$$

Using (3), we can now write the following fundamental inequalities:

Proposition 6. *Let $\mathcal{A} : \mathcal{I}^m \to M_n(K)$ be a fast gasket with coefficient $c_{\mathcal{A}}$ and let $J \in \mathcal{J}^m$ and $I, K \in \mathcal{I}^m$. Then, the following inequalities hold:*

$$n^{-s}\|A_{JK}\|^{-s}\zeta_{\mathcal{A}}(s) \le \zeta_{\mathcal{A}_{A_{JK}}}(s) \le c_{\mathcal{A}}^{-s}\|A_{JK}\|^{-s}\zeta_{\mathcal{A}}(s) \tag{4}$$

$$\zeta_{\mathcal{A}_{A_I}}(s) \ge \nu_{\mathcal{A}_{A_I}}(s) + n^{-s}\mu_{\mathcal{A}_{A_I}}(s)\zeta_{\mathcal{A}}(s) \tag{5}$$

$$\zeta_{\mathcal{A}_{A_I}}(s) \le \nu_{\mathcal{A}_{A_I}}(s) + c_{\mathcal{A}}^{-s}\mu_{\mathcal{A}_{A_I}}(s)\zeta_{\mathcal{A}}(s), \tag{6}$$

where $\nu_{\mathcal{A}_M}(s) = \sum_{D\in\mathcal{D}^m} \|A_D M\|^{-s}$ *and* $\mu_{\mathcal{A}_M}(s) = \sum_{J\in\mathcal{J}^m} \|A_J M\|^{-s}$.

Proof. The left and right sides of (4) are a direct consequence, respectively, of (2) and of the definition of fast gasket. The starting point to prove inequalities (5,6) is (3), from which we get

$$\zeta_{\mathcal{A}_{A_I}}(s) = \nu_{\mathcal{A}_{A_I}}(s) + \sum_{J\in\mathcal{J}^m} \zeta_{\mathcal{A}_{A_{JI}}}(s).$$

Applying the right side of (4) to the summation above, we get that

$$\sum_{J\in\mathcal{J}^m} \zeta_{\mathcal{A}_{A_{JI}}}(s) \le \sum_{J\in\mathcal{J}^m} c_{\mathcal{A}}^{-s}\|A_{JI}\|^{-s}\zeta_{\mathcal{A}}(s) = c_{\mathcal{A}}^{-s}\mu_{\mathcal{A}_{A_I}}(s)\zeta_{\mathcal{A}}(s).$$

Hence (6) follows and analogously it is proven (5). □

Remark 1. *The function $\nu_{\mathcal{A}}$ has the same complexity as the zeta functions of 1-gaskets since $\nu_{\mathcal{A}}(s) = \sum_{1\le i\le m} \zeta_{\mathcal{A}_i}(s)$, where $\mathcal{A}_i = \langle A_i \rangle$ are the 1-gaskets generated by the m generators of $\mathcal{A}$. In particular, by Proposition 2, this means that, unlike $\zeta_{\mathcal{A}}(s)$, even in the case $m > 1$ the series $\nu_{\mathcal{A}}(s)$ converges for some finite $s > 0$ for every gasket $\mathcal{A}$. The main idea for proving Theorem 1 is to exploit this fact to find upper and lower bounds for $\zeta_{\mathcal{A}}$ using the much simpler series $\nu_{\mathcal{A}}$.*

Remark 2. *For every s, the series $\mu_{\mathcal{A}}(s)$ and $\nu_{\mathcal{A}}(s)$ converge or diverge together. Indeed, if $J = iD \in \mathcal{J}^m$, then $J' = D \in \mathcal{D}^m$ and*

$$\frac{\|A_D\|}{n\|A_i^{-1}\|} \le \|A_{iD}\| \le n\|A_i\|\|A_D\|,$$

so that

$$(m-1)n^{-s}\min_{1\le i\le m}\|A_i\|^{-s}\nu_{\mathcal{A}}(s)\le\mu_{\mathcal{A}}(s)\le(m-1)n^{s}\max_{1\le i\le m}\|A_i^{-1}\|^{s}\nu_{\mathcal{A}}(s).$$

For similar reasons, $\nu_{\mathcal{A}_M}(s)$ and $\mu_{\mathcal{A}_M}(s)$ converge or diverge together with $\nu_{\mathcal{A}}(s)$ for every $M\in GL_n(K)$.

Let us now examine closely the two inequalities (5,6) for $I=0$. The first one becomes

$$\zeta_{\mathcal{A}}(s)\ge\nu_{\mathcal{A}}(s)+n^{-s}\mu_{\mathcal{A}}(s)\zeta_{\mathcal{A}}(s)$$

Since we are going to use this inequality to get lower bounds for $\zeta_{\mathcal{A}}(s)$, we can proceed without loss of generality by assuming that $\zeta_{\mathcal{A}}(s)<\infty$. Then, for $n^{-s}\mu_{\mathcal{A}}(s)<1$, we get that $\zeta_{\mathcal{A}}(s)\ge\frac{\nu_{\mathcal{A}}(s)}{1-n^{-s}\mu_{\mathcal{A}}(s)}$. Analogously, the right one becomes

$$\zeta_{\mathcal{A}}(s)\le\nu_{\mathcal{A}}(s)+c_{\mathcal{A}}^{-s}\mu_{\mathcal{A}}(s)\zeta_{\mathcal{A}}(s).$$

This time though, since we aim at using this inequality to provide upper bounds to the zeta function, we need to truncate the infinite series to a finite sum, in order to ensure that we are dealing with finite numbers. A natural recursive definition, inspired by the structure of (3), is the following:

$$\begin{array}{rcl}\zeta^0_{\mathcal{A}_M}(s)&=&\nu^0_{\mathcal{A}_M}(s)\\ \zeta^\ell_{\mathcal{A}_M}(s)&=&\nu^\ell_{\mathcal{A}_M}(s)+\displaystyle\sum_{\substack{J\in\mathcal{J}^m\\|J|\le\ell+1}}\zeta^{\ell+1-|J|}_{\mathcal{A}_{A_JM}}(s),\ \ell\ge1\end{array},$$

where $\nu^\ell_{\mathcal{A}_M}(s)=\sum_{D\in\mathcal{D}^m,|D|\le\ell}\|A_DM\|^{-s}$.

Proposition 7. *Let the sets $\mathcal{P}^m_\ell\subset\mathcal{I}^m$ be defined recursively as follows:*

$$\begin{array}{rcl}\mathcal{P}^m_0&=&\mathcal{D}^m_0,\\ \mathcal{P}^m_\ell&=&\left[\displaystyle\bigcup_{i=0}^{\ell}\mathcal{D}^m_i\right]\bigcup\left[\displaystyle\bigcup_{i=2}^{\ell+1}\mathcal{P}^m_{\ell+1-i}\cdot J^m_i\right],\ \ell\ge1.\end{array}$$

Then, the following properties hold:

1. *$\mathcal{P}^m_\ell\subset\mathcal{P}^m_{\ell+1}$ for all $\ell\ge0$;*
2. *$\displaystyle\bigcup_{\ell\ge0}\mathcal{P}^m_\ell=\mathcal{I}^m$;*

3. $\zeta^{\ell}_{\mathcal{A}_M}(s) = \sum_{I\in\mathcal{P}^m_\ell} \|A_I M\|^{-s}$.

Proof. 1. We prove this by induction. Clearly $\mathcal{P}^m_0 \subset \mathcal{P}^m_1$. Now, assume that $\mathcal{P}^m_k \subset \mathcal{P}^m_{k+1}$ for all $k \leq \ell-1$ and let $I \in \mathcal{P}^m_\ell$. Then, either $I \in \mathcal{D}^m_i$ for some $0 \leq i \leq \ell$, and therefore clearly $I \in \mathcal{P}^m_{\ell+1}$, or $I = PJ$ with $J \in \mathcal{J}^m_{\ell+1}$ and $P \in \mathcal{P}^m_{\ell+1-|J|}$. In this case, we have that $\ell+1-|J| \leq \ell-1$, since every element of $\mathcal{J}^m$ has at least rank two and therefore, by the inductive hypothesis, $\mathcal{P}^m_{\ell+1-|J|} \subset \mathcal{P}^m_{\ell+1-|J|+1}$. Hence, by definition, $I \in \mathcal{P}^m_{\ell+2-|J|} \cdot J \subset \mathcal{P}^m_{\ell+1}$.

2. Notice, first of all, that every index $I \in \mathcal{I}^m$ either belongs to $\mathcal{D}^m$ (and so to some $\mathcal{P}^m_\ell$) or it can be factored out as a product $I = D_0 J_1 \cdots J_k$, with $J_i \in \mathcal{J}^m$ and $D_0 \in \mathcal{D}^m$ (possibily $D_0 = 0$). This factorization simply consists in singling out the patterns of the form $i_1 \neq i_2 = i_3 = \cdots = i_p$ inside I starting from the right and is clearly unique. In case I's two leftmost indices are equal, then also a non-trivial $D_0 \in \mathcal{D}^m$ will appear in the decomposition. Now, let $j_i = |J_i| \geq 2$ and $d = |D_0| \geq 0$. By construction, we have that $D_0 \in \mathcal{P}^m_d$, $D_0 J_1 \in \mathcal{P}^m_d \mathcal{J}^m_{j_1} \subset \mathcal{P}^m_{d+j_1-1}$, $D_0 J_1 J_2 \in \mathcal{P}^m_{d+j_1-1} \mathcal{J}^m_{j_2} \subset \mathcal{P}^m_{d+j_1+j_2-2}$ and so on until we get $I \in \mathcal{P}^m_{|I|-k}$.

3. Let us write $\zeta^{\ell}_{\mathcal{A}_M}(s) = \sum_{I\in\mathcal{G}^\ell} \|A_I M\|^{-s}$. Then, if $I \in \mathcal{G}^\ell$ either $\|A_I M\|^{-s}$ appears in $\nu^{\ell}_{\mathcal{A}_M}(s)$, in which case $I \in \mathcal{D}^m$, $|I| \leq \ell$, by definition, or in some of the $\zeta^{\ell+1-|J|}_{\mathcal{A}_{A_J M}}(s)$, in which case $I = KJ$ with $K \in \mathcal{G}^{\ell+1-|J|}$. This is exactly the rule that defines recursively the $\mathcal{P}^m_\ell$. Since we also have, by the definition of $\zeta^0_{\mathcal{A}_M}(s)$, that $\mathcal{G}^0 = \mathcal{D}^m_0$, it follows that $\mathcal{G}^\ell = \mathcal{P}^m_\ell$. □

Corollary 2. *Let $\mathcal{A}$ be a fast gasket. Then, the $\zeta^{\ell}_{\mathcal{A}_M}(s)$ satisfy the following properties:*

$$\zeta^{\ell}_{\mathcal{A}_M}(s) \leq \zeta^{\ell+1}_{\mathcal{A}_M}(s) \text{ for all } \ell \geq 0; \tag{7}$$

$$\lim_{\ell\to\infty} \zeta^{\ell}_{\mathcal{A}_M}(s) = \zeta_{\mathcal{A}_M}(s); \tag{8}$$

$$n^{-s}\|A_J\|^{-s}\zeta^{\ell}_{\mathcal{A}}(s) \leq \zeta^{\ell}_{\mathcal{A}_{A_J}}(s) \leq c^{-s}_{\mathcal{A}}\|A_J\|^{-s}\zeta^{\ell}_{\mathcal{A}}(s); \tag{9}$$

$$\zeta^{\ell}_{\mathcal{A}_{A_I}}(s) \leq \nu_{\mathcal{A}_{A_I}}(s) + c^{-s}_{\mathcal{A}}\mu_{\mathcal{A}_{A_I}}(s)\zeta^{\ell}_{\mathcal{A}}(s). \tag{10}$$

Proof. (7,8) are a direct consequence of points 1. and 2. of the previous proposition. (9) is a direct consequence of (1) (left) and of the definition of fast gasket (right). Using the rhs of (9) and then (7), we get that

$$\zeta^{\ell}_{\mathcal{A}_M}(s) \leq \nu^{\ell}_{\mathcal{A}_M}(s) + c^{-s}_{\mathcal{A}}\mu^{\ell}_{\mathcal{A}_M}(s)\zeta^{\ell}_{\mathcal{A}}(s),$$

where $\mu^\ell_{\mathcal{A}_M}(s) = \sum_{J\in\mathcal{J}^m,|J|\leq\ell+1}\|A_J M\|^{-s}$. From this, (10) follows after setting $M = A_I$ and thanks to the monotonicity in ℓ of $\nu^\ell_{\mathcal{A}_M}(s)$ and $\mu^\ell_{\mathcal{A}_M}(s)$. $\square$

In particular, for $I = 0$ we get $\zeta^\ell_{\mathcal{A}}(s) \leq \nu_{\mathcal{A}}(s) + c_{\mathcal{A}}^{-s}\mu_{\mathcal{A}}(s)\zeta^\ell_{\mathcal{A}}(s)$. Hence, when $\mu_{\mathcal{A}}(s) < c^s_{\mathcal{A}}$, we have that $\zeta^\ell_{\mathcal{A}}(s) \leq \nu_{\mathcal{A}}(s)/(1 - c_{\mathcal{A}}^{-s}\mu_{\mathcal{A}}(s))$ for all ℓ and therefore, finally, $\zeta_{\mathcal{A}}(s) \leq \frac{\nu_{\mathcal{A}}(s)}{1-c_{\mathcal{A}}^{-s}\mu_{\mathcal{A}}(s)}$ when $c_{\mathcal{A}}^{-s}\mu_{\mathcal{A}}(s) < 1$.

With this, we proved the following:

Lemma 2. *Let $\mathcal{A}$ be a fast gasket with coefficient $c_{\mathcal{A}}$. Then*

$$\zeta_{\mathcal{A}}(s) \geq \frac{\nu_{\mathcal{A}}(s)}{1 - n^{-s}\mu_{\mathcal{A}}(s)} \text{ for all } s \text{ such that } \mu_{\mathcal{A}}(s) < n^s \tag{11}$$

and

$$\zeta_{\mathcal{A}}(s) \leq \frac{\nu_{\mathcal{A}}(s)}{1 - c_{\mathcal{A}}^{-s}\mu_{\mathcal{A}}(s)} \text{ for all } s \text{ such that } \mu_{\mathcal{A}}(s) < c^s_{\mathcal{A}}. \tag{12}$$

This Lemma implies immediately our first main result (Theorem 1) in particular cases, e.g. when all generators of $\mathcal{A}$ have non-negative coefficients and norms larger than the maximum between 1 and the inverse of the coefficient $c_{\mathcal{A}}$ of the gasket.

Consider indeed (11). Under these assumptions, the function $g_{\mathcal{A}}(s) = n^{-s}\mu_{\mathcal{A}}(s)$ is clearly monotonically decreasing with s and both its domain and its image equal $(0, \infty)$. In particular, the set $g_{\mathcal{A}}(s) < 1$ is not empty and therefore (11) holds for $s > s_g$, with $s_g = g^{-1}_{\mathcal{A}}(1)$. When we let $s \to s_g^+$, the right hand side of (11) goes to infinity, so that $\zeta_{\mathcal{A}}(s_g)$ diverges, namely $s_{\mathcal{A}} \geq s_g$.

Now, consider (12). The function $f_{\mathcal{A}}(s) = c^{-s}\mu_{\mathcal{A}}(s)$ satisfies the same properties listed above for $g_{\mathcal{A}}$. Let $s_f = f^{-1}_{\mathcal{A}}(1)$. Then $f_{\mathcal{A}}(s) < 1$ for $s > s_f$ so that (11) holds. This proves that $\zeta_{\mathcal{A}}(s) < \infty$ for all $s > s_f$, namely $s_f \geq s_{\mathcal{A}}$.

This simple argument not only implies that $s_{\mathcal{A}}$ is finite, but also provides for it non-trivial lower and upper bounds. In order to obtain a similar result in full generality we need to refine (12). This shall lead us to refine also (11) and to generate a pair of sequences converging to $s_{\mathcal{A}}$ from the left and from the right at logarithmic speed. The idea is to apply over and over recursively first the inequalities (5,6) and then the inequality (4) to the truncated zeta function.

The starting point is the sequence of sets of multi-indices $\mathcal{Q}_{\mathcal{A},k}$ built as follows. We define $\mathcal{Q}^0_{\mathcal{A},k} = \mathcal{J}^m$. Then we consider the sets recursively defined as

$$\mathcal{Q}^\ell_{\mathcal{A},k} = \left\{J \in \mathcal{Q}^{\ell-1}_{\mathcal{A},k} \,|\, \|A_J\| > k\right\} \bigcup \mathcal{J}^m \cdot \left\{J \in \mathcal{J}^m \,|\, J \in \mathcal{Q}^{\ell-1}_{\mathcal{A},k}, \|A_J\| \le k\right\}$$

with $\ell \ge 1$.

Proposition 8. *For every gasket $\mathcal{A}$ and every $k > 0$ there exists a $\bar\ell$ such that $\mathcal{Q}^{\ell'}_{\mathcal{A},k} = \mathcal{Q}^{\ell'+1}_{\mathcal{A},k}$ for every $\ell' \ge \bar\ell$.*

Proof. The sole effect of the algorithm is replacing all indices $I \in \mathcal{Q}^\ell_{\mathcal{A},k}$, corresponding to matrices A_I such that $\|A_I\| \le k$, with indices of higher order. The set $\mathcal{Q}^{\ell+1}_{\mathcal{A},k}$ thus obtained might still contain indices corresponding to matrices with norm not larger than k, but in a finite number of steps all such indices will disappear because, by definition of gasket, there is only a finite amount of them. Hence, there exists a finite $\bar\ell$ such that $\|A_I\| > k$ for all $I \in \mathcal{Q}^{\bar\ell}_{\mathcal{A},k}$. Clearly the algorithm leaves unchanged all sets $\mathcal{Q}^{\ell'}_{\mathcal{A},k}$ with $\ell' \ge \bar\ell$. ☐

Definition 4. *We use the notation $\mathcal{Q}_{\mathcal{A},k} = \mathcal{Q}^{\bar\ell}_{\mathcal{A},k}$ and, correspondingly,*

$$f^\ell_{\mathcal{A},k}(s) = c_{\mathcal{A}}^{-s} \sum_{J \in \mathcal{Q}^\ell_{\mathcal{A},k}} \|A_J\|^{-s}, \qquad f_{\mathcal{A},k}(s) = c_{\mathcal{A}}^{-s} \sum_{J \in \mathcal{Q}_{\mathcal{A},k}} \|A_J\|^{-s} \tag{13}$$

$$g^\ell_{\mathcal{A},k}(s) = n^{-s} \sum_{J \in \mathcal{Q}^\ell_{\mathcal{A},k}} \|A_J\|^{-s}, \qquad g_{\mathcal{A},k}(s) = n^{-s} \sum_{J \in \mathcal{Q}_{\mathcal{A},k}} \|A_J\|^{-s}. \tag{14}$$

Note that $f_{\mathcal{A},0}(s) = f_{\mathcal{A}}(s)$ and $g_{\mathcal{A},0}(s) = g_{\mathcal{A}}(s)$. Next proposition, though, shows that, in the general case, for k big enough, $f_{\mathcal{A},k}$ and $g_{\mathcal{A},k}$ have a nicer behaviour than $f_{\mathcal{A}}$ and $g_{\mathcal{A}}$.

Proposition 9. *For every fast gasket $\mathcal{A}$, there exists a $\bar k$ such that, for all $k > \bar k$, $f_{\mathcal{A},k}(s)$ and $g_{\mathcal{A},k}(s)$ are strictly decreasing continuous functions of s defined in some non-empty positive half-line $(\gamma_{\mathcal{A},k}, \infty)$ and have image $(0, \infty)$.*

Proof. By construction, $f_{\mathcal{A},k}(s)$ and $g_{\mathcal{A},k}(s)$ are proportional to each other and, respectively with constants $c_{\mathcal{A}}^{-s}$ and n^{-s}, to the sum of a finite number of functions $\mu^{(k)}_{\mathcal{A}_{A_I}}$ defined as the series $\mu_{\mathcal{A}_{A_I}}$ from which all terms with norm smaller than k have been subtracted.

By Remarks 1 and 2 then

$$\mu^{(k)}_{\mathcal{A}_{A_I}}(s) \leq (m-1)n^s \max_{1\leq i\leq m} \|A_i^{-1}\|^s \sum_{I\in\mathcal{G}} \nu^{(k)}_{\mathcal{A}_{A_I}}(s), \tag{15}$$

where $\mathcal{G} \subset \mathcal{I}^m$ is some finite set of indices and the series $\nu^{(k)}_{\mathcal{A}_{A_I}}$ is equal to $\nu_{\mathcal{A}_{A_I}}$ minus those terms with indices $D \in \mathcal{D}^m$ such that $\|A_{JI}\| \leq k$ for all $J \in \mathcal{J}^m$ with $J' = D$. In particular then, $f_{\mathcal{A},k}(s)$ and $g_{\mathcal{A},k}(s)$ are bounded above and below by a finite sum of series $\nu^{(k)}_{\mathcal{A}_{A_I}}(s)$ so that, by the same arguments used in Section 2.3, they are finite on some connected non-empty interval $(\gamma_{\mathcal{A},k}, \infty)$.

Now, let $\bar{k} = n^2 \max_{1\leq i\leq m} \|A_i^{-1}\| \max_{1\leq i\leq m} \|A_i\|$. For every $k > \bar{k}$ we have that, for every $J \in \mathcal{J}^m$ such that the term $\|A_{J'I}\|^{-s}$ appears in $\nu^{(k)}_{\mathcal{A}_{A_I}}(s)$ and $\|A_{JI}\| > k$,

$$\|A_{J'I}\| \geq \frac{\|A_{JI}\|}{n\|A_i\|} > \frac{k}{n\|A_i\|} > n \max_{1\leq i\leq m} \|A_i^{-1}\|,$$

namely

$$\frac{n \max\limits_{1\leq i\leq m} \|A_i^{-1}\|}{\|A_{J'I}\|} \leq \frac{n^2 \max\limits_{1\leq i\leq m} \|A_i^{-1}\| \max\limits_{1\leq i\leq m} \|A_i\|}{k} < 1.$$

This means that all summands in the series in the right hand side of (15) are s-powers of numbers uniformly bounded from above by some number strictly smaller than 1 and therefore they are strictly decreasing with s and their image equals $(0, \infty)$. By (15), the same holds for $f_{\mathcal{A},k}(s)$ and $g_{\mathcal{A},k}(s)$. $\square$

Theorem 1. *Let $\mathcal{A}$ be a fast gasket. Then $0 < s_{\mathcal{A}} < \infty$ and both $s_{g,k} = g^{-1}_{\mathcal{A},k}(1)$ and $s_{f,k} = f^{-1}_{\mathcal{A},k}(1)$ are uniquely defined and converge to $s_{\mathcal{A}}$, respectively from left and right, with speed at least logarithmic as $k \to \infty$.*

Proof. We start by showing that

$$\zeta_{\mathcal{A}}(s) \geq h_{\mathcal{A},k}(s) + g_{\mathcal{A},k}(s)\zeta_{\mathcal{A}}(s) \tag{16}$$

and

$$\zeta_{\mathcal{A}}(s) \leq h_{\mathcal{A},k}(s) + f_{\mathcal{A},k}(s)\zeta_{\mathcal{A}}(s) \tag{17}$$

for every k, where $h_{\mathcal{A},k}$ is some positive continuous function. We prove it in detail for the most complicated case, namely the second one, where we need to use the partial sums $\zeta^{\ell}_{\mathcal{A}}$. First, we notice that (10) writes as

$$\zeta^{\ell}_{\mathcal{A}_{A_I}}(s) \leq h^0_{\mathcal{A}_{A_I},k} + f^0_{\mathcal{A}_{A_I},k}(s)\zeta^{\ell}_{\mathcal{A}}(s) \tag{18}$$

after putting $h^0_{\mathcal{A}A_I,k}(s) = \nu_{\mathcal{A}A_I}(s)$. Now, we write the definition of $\zeta^\ell_{\mathcal{A}A_I}(s)$ splitting the last term in two

$$\zeta^\ell_{\mathcal{A}A_I}(s) = \nu^\ell_{\mathcal{A}A_I}(s) + \sum_{\substack{J\in\mathcal{J}^m_{\ell+1}\\ \|A_J\|\le k}} \zeta^{\ell+1-|J|}_{\mathcal{A}A_{JI}}(s) + \sum_{\substack{J\in\mathcal{J}^m_{\ell+1}\\ \|A_J\|> k}} \zeta^{\ell+1-|J|}_{\mathcal{A}A_{JI}}(s). \tag{19}$$

By using the monotonicity in ℓ of $\nu^\ell_{\mathcal{A}A_I}$ on the first term and applying (10) to the second term and (9) to the third, we get that

$$\zeta^\ell_{\mathcal{A}A_I}(s) \le \nu_{\mathcal{A}A_I}(s) + \sum_{\substack{J\in\mathcal{J}^m\\ \|A_J\|\le k}} \left[\nu^\ell_{\mathcal{A}A_{JI}}(s) + c_{\mathcal{A}}^{-s}\mu_{\mathcal{A}A_{JI}}(s)\zeta^\ell_{\mathcal{A}}(s)\right] + c_{\mathcal{A}}^{-s}\sum_{\substack{J\in\mathcal{J}^m\\ \|A_J\|> k}} \|A_{JI}\|^{-s}\zeta^\ell_{\mathcal{A}}(s)$$

$$= h^1_{\mathcal{A}A_I,k}(s) + c_{\mathcal{A}}^{-s}\sum_{L\in\mathcal{Q}^1_{\mathcal{A},k}} \|A_L\|^{-s}\zeta^\ell_{\mathcal{A}}(s) = h^1_{\mathcal{A}A_I,k}(s) + f^1_{\mathcal{A}A_I,k}(s)\zeta^\ell_{\mathcal{A}}(s),$$

where we set $h^1_{\mathcal{A}A_I,k}(s) = h^0_{\mathcal{A}A_I,k}(s) + \sum_{\substack{J\in\mathcal{J}^m\\ \|A_J\|\le k}} \nu_{\mathcal{A}A_{JI}}(s)$.

By repeating recursively this procedure, we get that

$$\zeta^\ell_{\mathcal{A}}(s) \le h^r_{\mathcal{A},k}(s) + f^r_{\mathcal{A},k}(s)\zeta^\ell_{\mathcal{A}}(s)$$

for all $r \ge 0$. Since the $\mathcal{Q}^r_{\mathcal{A},k}$ stabilize eventually into the $\mathcal{Q}_{\mathcal{A},k}$ and the ζ^ℓ are monotonically increasing in ℓ, we proved that

$$\zeta^\ell_{\mathcal{A}}(s) \le h_{\mathcal{A},k}(s) + f_{\mathcal{A},k}(s)\zeta_{\mathcal{A}}(s).$$

for every ℓ and therefore (17) follows; (16) is proved analogously.

By Proposition 9, for k big enough the points $s_{f,k} = f^{-1}_{\mathcal{A},k}(1)$ and $s_{g,k} = g^{-1}_{\mathcal{A},k}(1)$ are uniquely defined and strictly between 0 and ∞. As already pointed out right after Lemma 2, inequalities (16) and (17) imply that $s_{g,k} \le s_{\mathcal{A}} \le s_{f,k}$ for every such k. In particular, we have that $0 < s_{\mathcal{A}} < \infty$, since every $s_{f,k}$ is finite and every $s_{g,k}$ is larger than 0.

In order to prove the last part of the theorem, we note that $g_{\mathcal{A},k}(s) = c^s_{\mathcal{A}} n^{-s} f_{\mathcal{A},k}(s)$ and that, for any $s' > 0$,

$$f_{\mathcal{A},k}(s) > (c_{\mathcal{A}}k)^{s'} c_{\mathcal{A}}^{-s-s'} \sum_{J\in\mathcal{Q}_{\mathcal{A},k}} \|A_J\|^{-s-s'} = (c_{\mathcal{A}}k)^{s'} f_{\mathcal{A},k}(s+s').$$

Hence

$$1 = g_{\mathcal{A},k}(s_{g,k}) = c_{\mathcal{A}}^{s_{g,k}} n^{-s_{g,k}} f_{\mathcal{A},k}(s_{g,k}) >$$

$$> c_{\mathcal{A}}^{s_{g,k}} n^{-s_{g,k}} (c_{\mathcal{A}} k)^{s_{f,k}-s_{g,k}} f_{\mathcal{A},k}(s_{g,k}+s_{f,k}-s_{g,k}) = c_{\mathcal{A}}^{s_{f,k}} n^{-s_{g,k}} k^{s_{f,k}-s_{g,k}},$$

so that

$$0 > s_{f,k}(\log k + \log c_{\mathcal{A}}) - s_{g,k}(\log k + \log n)$$

and finally

$$0 \leq s_{f,k} - s_{g,k} < s_{g,k}\left(\frac{\log k + \log n}{\log k + \log c_{\mathcal{A}}} - 1\right) \leq s_{\mathcal{A}} \frac{\log n - \log c_{\mathcal{A}}}{\log k + \log c_{\mathcal{A}}}.$$

□

2.5 An alternate characterization of $s_{\mathcal{A}}$

Our second main result shows that the exponent $s_{\mathcal{A}}$ of a gasket $\mathcal{A}$ can also be extracted from the asymptotics of the partial sums of the $\|A_I\|^{-s}$ over same-rank multi-indices, namely from the sequence of functions $\zeta_{\mathcal{A},k}(s) = \sum_{I\in\mathcal{I}_k^m} \|A_I\|^{-s}$.

Lemma 3. *The sequence of analytical log-convex monotonically decreasing functions $\zeta_{\mathcal{A},k}^{1/k}(s)$ converges pointwise, for every $s \in [0,\infty)$, to a bounded continuous log-convex monotonically decreasing function $\xi_{\mathcal{A}}(s)$.*

Proof. Since $\mathcal{A}$ is a gasket, from some $\bar{k}$ on each A_I, $|I| \geq \bar{k}$, has norm larger than 1. Hence, the $\zeta_{\mathcal{A},k}^{1/k}(s)$, $k \geq \bar{k}$, are analytical log-convex monotonically decreasing functions, because every summand $\|A_I\|^{-s}$ of the $\zeta_{\mathcal{A},k}$ satisfies those properties and so does every finite or infinite (converging) sum and positive power of them.

In order to prove the convergence of the sequence, we can replace, without loss of generality, the max norm in the expression of the $\zeta_{\mathcal{A},k}$ (by abuse of notation we will denote the new functions still by $\zeta_{\mathcal{A},k}$) with any submultiplicative norm $\|\cdot\|'$ and notice that, since $\|A_{IJ}\|' \leq \|A_I\|'\|A_J\|'$, then $\zeta_{\mathcal{A},k+k'}(s) \geq \zeta_{\mathcal{A},k}(s)\zeta_{\mathcal{A},k'}(s)$. It follows that, for every s, the sequence $\zeta_{\mathcal{A},k}^{1/k}(s)$ can have only one accumulation point and this point must be equal to $\sup_{k\in\mathbb{N}} \zeta_{\mathcal{A},k}^{1/k}(s)$. The main point is that, for every element $\zeta_{\mathcal{A},k_0}^{1/k_0}(s)$, almost all other elements of the sequence are not smaller than $\zeta_{\mathcal{A},k_0}^{1/k}(s) - \varepsilon$ for every $\varepsilon > 0$. Indeed, if $k = Nk_0$, then immediately $\zeta_{\mathcal{A},k}^{1/k}(s) \geq \left[\zeta_{\mathcal{A},k_0}^{N}(s)\right]^{1/k_0} = \zeta_{\mathcal{A},k_0}^{1/k_0}(s)$, while if $k = Nk_0 + \ell$, with $1 \leq \ell \leq k_0 - 1$, then $\zeta_{\mathcal{A},k}(s) \geq \zeta_{\mathcal{A},k_0}^{N}(s)\zeta_{\mathcal{A},\ell}(s)$, so that

$$\zeta_{\mathcal{A},k}^{1/k}(s) \geq \zeta_{\mathcal{A},k_0}^{\frac{N}{Nk_0+\ell}}(s)\zeta_{\mathcal{A},\ell}^{\frac{1}{Nk_0+\ell}}(s) = \left[\zeta_{\mathcal{A},k_0}^{1/k_0}(s)\right]^{\frac{1}{1+\ell/(Nk_0)}} \zeta_{\mathcal{A},\ell}^{\frac{1}{Nk_0+\ell}}(s).$$

Since there are only a finite number of possible values of ℓ, for every $\varepsilon' > 0$ we can find a N big enough such that both $\left|\left[\zeta^{1/k_0}_{\mathcal{A},k_0}(s)\right]^{\frac{1}{1+\ell/(Nk_0)}} - \zeta^{1/k_0}_{\mathcal{A},k_0}(s)\right| < \varepsilon'$ and $\left|\zeta^{\frac{1}{Nk_0+\ell}}_{\mathcal{A},\ell}(s) - 1\right| < \varepsilon'$ hold for all ℓ. Hence

$$\zeta^{1/k}_{\mathcal{A},k}(s) \geq \zeta^{1/k_0}_{\mathcal{A},k_0}(s) - \varepsilon' \left(\zeta^{1/k_0}_{\mathcal{A},k_0}(s) + 1 - \varepsilon'\right) \geq \zeta^{1/k_0}_{\mathcal{A},k_0}(s) - \varepsilon$$

for small enough ε'.

That $\xi_{\mathcal{A}}(s) = \sup_{k\in\mathbb{N}} \zeta^{1/k}_{\mathcal{A},k}(s)$ is finite for all s it is clear from the fact that all $\zeta^{1/k}_{\mathcal{A},k}(s)$ are positive decreasing functions bounded by $\zeta^{1/k}_{\mathcal{A},k}(0) = m$. □

Theorem 2. *The function $\xi_{\mathcal{A}}$ satisfies the following properties:*

1. *$\xi_{\mathcal{A}}(s) > 1$ for $s < s_{\mathcal{A}}$;*
2. *$\xi_{\mathcal{A}}(s_{\mathcal{A}}) = 1$;*
3. *$\xi_{\mathcal{A}}(s) < 1$ for $s > s_{\mathcal{A}}$, if $\mathcal{A}$ is a hyperbolic gasket;*
4. *$\xi_{\mathcal{A}}(s) = 1$ for $s > s_{\mathcal{A}}$, if $\mathcal{A}$ is a parabolic fast gasket.*

Proof. Directly from the n-th root test, we get that $\xi_{\mathcal{A}}(s) \geq 1$ for $s < s_{\mathcal{A}}$ and $\xi_{\mathcal{A}}(s) \leq 1$ for $s > s_{\mathcal{A}}$, so that in particular $\xi_{\mathcal{A}}(s_{\mathcal{A}}) = 1$.

Assume first that $\mathcal{A}$ is hyperbolic, so that there exist constants $\alpha > 1$ and $K > 0$ such that $\|A_I\| \geq K\alpha^{|I|}$ for every $I \in \mathcal{I}^m$. Hence

$$\frac{d}{ds} \ln \zeta^{1/k}_k(s) = -\frac{1}{k} \frac{\sum_{|I|=k} \left(\|A_I\|^{-s} \ln \|A_I\|\right)}{\sum_{|I|=k} \|A_I\|^{-s}} \leq -\ln\alpha - \frac{\ln K}{k},$$

namely for every $\varepsilon > 0$ we can find a $\alpha' > 1$, with $|\alpha - \alpha'| \leq \varepsilon$, and a $\bar{k} > 0$ such that $(\ln \zeta^{1/k}_k(s))' \leq -\ln\alpha'$ for all $k \geq \bar{k}$. Since $\ln \zeta^{1/k}_k(s_{\mathcal{A}}) = 0$, this means that, for every $k \in \mathbb{N}$ and $s > 0$, $\ln \zeta^{1/k}_k(s_{\mathcal{A}} + s) \leq -s\ln\alpha'$ and $\ln \zeta^{1/k}_k(s_{\mathcal{A}} - s) \geq s\ln\alpha'$, namely $\zeta^{1/k}_k(s) \geq (\alpha')^s > 1$ at the left of $s_{\mathcal{A}}$ and $\zeta^{1/k}_k(s) \leq (\alpha')^{-s} < 1$ at its right. Since this is true for almost all k, the same properties hold for $\xi_{\mathcal{A}}$.

Assume now that $\mathcal{A}$ is fast parabolic and that $s_{\mathcal{A}} < \infty$ (e.g. in case that $\mathcal{A}$ is fast). In this case, the sequence $a_k = \min_{|I|=k}\{\|A_I\|\}$ grows polynomially and therefore, for $s > s_{\mathcal{A}}$,

$$1 \geq \zeta^{1/k}_k(s) \geq a_k^{-s/k} \xrightarrow{k\to\infty} 1, \text{ i.e. } \xi_{\mathcal{A}}(s) = \lim_{k\to\infty} \zeta^{1/k}_k(s) = 1.$$

Let now $s < s_{\mathcal{A}}$. Analogously to (4) and (3), we have the inequality

$$\zeta_{\mathcal{A}_{A_I},k}(s) \geq \frac{1}{n^s \|A_I\|^s} \zeta_{\mathcal{A},k}(s) \tag{20}$$

and we can re-write $\zeta_{\mathcal{A}_{A_I},k}$ as follows:

$$\zeta_{\mathcal{A}_{A_I},k}(s) = \sum_{\substack{D \in \mathcal{D}^m \\ |D|=k}} \zeta_{\mathcal{A}_{A_{DI}},0}(s) + \sum_{\substack{J \in \mathcal{J}^m \\ |J| \leq k}} \zeta_{\mathcal{A}_{A_{JI}},k-|J|}(s). \tag{21}$$

By repeating step by step arguments used in Theorem 1, we get that

$$\zeta_{\mathcal{A}_{A_I},k}(s) \geq H_{\mathcal{A},\kappa}(s) + \sum_{2 \leq j \leq k} G^j_{\mathcal{A}_{A_I},\kappa}(s) \zeta_{\mathcal{A},k-j}(s),$$

for every $\kappa > 0$, where

$$G^j_{\mathcal{A}_{A_I},\kappa}(s) = n^{-s} \sum_{J \in \mathcal{Q}^m_{\mathcal{A},\kappa} \bigcap \mathcal{I}^m_j} \|A_{JI}\|^{-s}. \tag{22}$$

Now, consider the polynomials $p_k(x) = x^k - \sum_{2 \leq j \leq k} G^j_{\mathcal{A}_{A_I},\kappa}(s) x^{k-j}$, $k \in \mathbb{N}$. By Descartes' rule of signs, they all have a single positive root. Moreover, for every κ and $s < s_{g,\kappa}$, we can find a $\bar{k}$ big enough so that this root is larger than 1. Indeed, by (22), the finite sum $\sum_{2 \leq j \leq k} G^j_{\mathcal{A}_{A_I},\kappa}(s)$ is equal to the restriction of the series $g_{\mathcal{A}_{A_I},\kappa}(s)$ to the terms $\|A_I\|^{-s}$ with $|I| \leq k$ and, by the definition of $s_{g,\kappa}$, $g_{\mathcal{A}_{A_I},\kappa}(s) > 1$ for $s < s_{g,\kappa}$. Then $p_{\bar{k}}(1) < 0$ and, therefore, its only positive root σ must be larger than 1. Hence

$$\mu \stackrel{\text{def}}{=} \min_{0 \leq j \leq \bar{k}} \{\zeta_{\mathcal{A},j}(s) \sigma^{-j}\} = \inf_{0 \leq j \leq \infty} \{\zeta_{\mathcal{A},j}(s) \sigma^{-j}\},$$

which follows by induction as a consequence of the following observation:

$$\zeta_{\mathcal{A},\bar{k}+1}(s) \geq \sum_{2 \leq j \leq \bar{k}} G^j_{\mathcal{A}_{A_I},\kappa}(s) \zeta_{\mathcal{A},\bar{k}+1-j}(s) \geq \mu \sum_{2 \leq j \leq \bar{k}} G^j_{\mathcal{A}_{A_I},\kappa}(s) \sigma^{\bar{k}+1-j} \geq \mu \sigma^{\bar{k}+1}.$$

Finally then, $\xi_{\mathcal{A}}(s) = \lim_{j \to \infty} \zeta^{1/j}_{\mathcal{A},j}(s) \geq \sigma > 1$ for every $s \leq s_{g,\kappa}$. Since, by choosing big enough κ, we can get $s_{g,\kappa}$ as close as we please to $s_{\mathcal{A}}$, it follows that $\xi_{\mathcal{A}}(s) > 1$ for every $s < s_{\mathcal{A}}$. $\square$

Example 8. *Let $M_1, \dots, M_m$ be upper triangular matrices of the form*

$$M_i = \begin{pmatrix} \alpha_i & \beta_i \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{R}^+),$$

and assume that $\max_{1\le i\le m}\{\beta_i\} \le 1 - \max_{1\le i\le m}\{\alpha_i\}$. It is easy to prove by induction that, under this assumption, the non-zero off-diagonal term never gets larger than 1, so that $\|M_I\| = 1$ for every $I \in \mathcal{I}^m$.

Now, consider the gasket $\mathcal{A}$ generated by $A_i = \rho_i M_i$, $\rho_i > 1$. By the observation above, for every $I = i_1 \dots i_k$, we have that $\|A_I\| = \rho_{i_1} \cdots \rho_{i_k}$ and, therefore,

$$\zeta_{\mathcal{A},k} = \sum_{I\in\mathcal{I}^m_k} \|A_I\|^{-s} = \sum_{I\in\mathcal{I}^m_k} \rho_{i_1}^{-s} \cdots \rho_{i_k}^{-s} = \left(\rho_1^{-s} + \cdots + \rho_m^{-s}\right)^k .$$

Since $\mathcal{A}$ is clearly a hyperbolic gasket, by Theorem 2 its exponent $s_{\mathcal{A}}$ is the unique solution of the equation $\rho_1^{-s} + \cdots + \rho_m^{-s} = 1$. Similar, but more complicated, conditions can be found for upper triangular matrices in higher dimension.

2.6 Norm asymptotics of fast gaskets

Our third main result gives a further characterization of the exponent $s_{\mathcal{A}}$ of a gasket $\mathcal{A}$ in terms of the asymptotic properties of the function $N_{\mathcal{A}}(r)$ (see Definition 1).

Lemma 4. *Let $\mathcal{A} : \mathcal{I}^m \to M_n(K)$ be a semigroup homomorphism and $M \in GL_n(K)$. Then*

$$N_{\mathcal{A}_M}(r) > N_{\mathcal{A}}\left(\frac{r}{n\|M\|}\right). \tag{23}$$

Proof. Since $\|AM\| \le n\|A\|\|M\|$, we have that $\|A_I\| \le \frac{r}{n\|M\|} \implies \|A_I M\| \le r$, namely $\{A_I | \|A_I\| \le \frac{r}{n\|M\|}\} \subset \{A_I M | \|A_I M\| \le r\}$ □

Theorem 3. *Let $\mathcal{A} : \mathcal{I}^m \to M_n(K)$ be a hyperbolic or fast parabolic gasket. Then*

$$\lim_{r\to\infty} \frac{\log N_{\mathcal{A}_M}(r)}{\log r} = s_{\mathcal{A}}$$

for every $M \in GL_n(K)$.

Proof. Since $\|A\|/(n\|M^{-1}\|) \le \|AM\| \le n\|A\|\|M\|$, we can prove the theorem without loss of generality in the particular case $M = \mathbb{1}_n$.

$\boldsymbol{\limsup_{r\to\infty} \frac{\log N_{\mathcal{A}}(r)}{\log r} \le s_{\mathcal{A}}}$. Let $s > s_{\mathcal{A}}$. Then

$$\infty > \zeta_{\mathcal{A}}(s) > \sum_{\|A_I\|\le r} \|A_I\|^{-s} \ge \sum_{\|A_I\|\le r} r^{-s} = N_{\mathcal{A}}(r) r^{-s},$$

so that

$$s + \frac{\log \zeta_{\mathcal{A}}(s)}{\log r} > \frac{\log N_{\mathcal{A}}(r)}{\log r}$$

and therefore $\limsup_{r\to\infty} \frac{\log N_{\mathcal{A}}(r)}{\log r} \le s$. Since this is true for every $s > s_{\mathcal{A}}$, it follows at once that $\limsup_{r\to\infty} \frac{\log N_{\mathcal{A}}(r)}{\log r} \le s_{\mathcal{A}}$.

$\boldsymbol{\liminf_{r\to\infty} \frac{\log N_{\mathcal{A}}(r)}{\log r} \ge s_{\mathcal{A}}}$. From the elementary observation that

$$\{A_I | \|A_I\| \le r, I \in \mathcal{I}^m\} \supset \bigcup_{K\in\mathcal{I}^m_k} \{A_{IK} | \|A_{IK}\| \le r, I \in \mathcal{I}^m\}$$

and using (23), we get that, for every $k \in \mathbb{N}$,

$$N_{\mathcal{A}}(r) \ge \sum_{K\in\mathcal{I}^m_k} N_{\mathcal{A}_{A_K}}(r) \ge \sum_{K\in\mathcal{I}^m_k} N_{\mathcal{A}}\left(\frac{r}{n\|A_K\|}\right).$$

Assume now $s < s_{\mathcal{A}}$. Since $\mathcal{A}$ is, by hypothesis, either a hyperbolic or a fast parabolic gasket, by the definition of gasket and Theorem 2, we can always choose a k_0 such that $\|A_I\| > 1/n$ for $|I| \ge k_0$ and $\sum_{I\in\mathcal{I}^m_{k_0}} \|A_I\|^{-s} > n^s$.

Now, set $a_m = n \min_{I\in\mathcal{I}^m_{k_0}} \|A_I\|$ and $a_M = n \max_{I\in\mathcal{I}^m_{k_0}} \|A_I\|$, let $r_0 > 0$ be such that $N_{\mathcal{A}}(r_0) > 0$ and set $r_1 = a_M r_0$ and $r_i = a_m^{i-1} r_1$, $i \ge 2$. Similar to the proof of Theorem 2, we have by induction that,

$$M \stackrel{\text{def}}{=} \min_{r\in[r_0,r_1]} N_{\mathcal{A}}(r_0) r^{-s} = \inf_{r\in[r_0,\infty]} N_{\mathcal{A}}(r_0) r^{-s}.$$

Indeed, note first of all that $\lim_{i\to\infty} r_i = \infty$, since we chose k_0 so that $a_m > 1$. Assume now that $N_{\mathcal{A}}(r) \ge M r^s$ in $[r_0, r_i]$ and let $r \in [r_i, r_{i+1}]$. Then, for every $I \in \mathcal{I}^m_{k_0}$, we have that $r/(n\|A_I\|) \in [r_0, r_i]$, since

$$r_i = \frac{r_{i+1}}{a_m} \ge \frac{r}{n\|A_I\|} \ge \frac{r_i}{a_M} = a_m^{i-1} r_0 \ge r_0.$$

and, therefore,

$$N_{\mathcal{A}}(r) \geq \sum_{I \in \mathcal{I}_k^m} N_{\mathcal{A}}\left(\frac{r}{n\|A_I\|}\right) \geq \sum_{I \in \mathcal{I}_k^m} M\left[\frac{r}{n\|A_I\|}\right]^s \geq Mr^s.$$

Hence, it follows at once that

$$\frac{\log N_{\mathcal{A}}(r)}{\log r} \geq \frac{\log M}{\log r} + s$$

and, therefore, $\liminf_{r\to\infty} \frac{logN_{\mathcal{A}}(r)}{\log r} \geq s$. Since this is true for all $s < s_{\mathcal{A}}$, it follows that $\liminf_{r\to\infty} \frac{logN_{\mathcal{A}}(r)}{\log r} \geq s_{\mathcal{A}}$. □

Corollary 3. *Let $\{I_n\}$ be an ordering of all indices in $\mathcal{I}^m$ such that $\|A_{I_n}\| \leq \|A_{I_{n'}}\|$ for $n < n'$. Then*

$$\lim_{n\to\infty} \frac{\log \|A_{I_n}\|}{\log n} = \frac{1}{s_{\mathcal{A}}}.$$

Proof. This is a direct consequence of the fact that, for a given $r = \|A_{I_n}\|$, the number of matrices with norm not larger than r is $n = N(r)$. □

Corollary 4. *Let $r_{\mathcal{A}} = \lim_{k\to\infty} \max_{|I|=k} \|A_I\|^{1/k}$ be the* joint spectral radius *of the m generators of $\mathcal{A}$. Then $r_{\mathcal{A}} \geq m^{1/s_{\mathcal{A}}}$.*

Proof. Assume first that all terms A_I, $|I| = k$, have the same norm $N_k = \lambda^k$, so that $r_{\mathcal{A}} = \lim_{k\to\infty} N_k^{1/k} = \lambda$. Then $N_{\mathcal{A}}(N_k) = \frac{m^k-1}{m-1}$ and, therefore, $s_{\mathcal{A}} = \lim_{k\to\infty} \log N_{\mathcal{A}}(N_k)/\log N_k = 1/\log_m \lambda$, so that $r_{\mathcal{A}} = m^{1/s_{\mathcal{A}}}$. When not all the products of k generators of $\mathcal{A}$ have the same norm, it means that $N_{\mathcal{A}}(N_k)$ is larger than in the previous case and, consequently, $s_{\mathcal{A}}$ might increase. Hence, in general, $r_{\mathcal{A}} \geq m^{1/s_{\mathcal{A}}}$.

Now assume that N_k is polynomial in k. Then $s_{\mathcal{A}} = \infty$ and $r_{\mathcal{A}} = 1$, so even in this case the relation holds. □

Remark 3. *This inequality cannot be improved. Indeed, let $\mathcal{A}$ be the gasket generated by the two matrices $\rho M_1, \rho M_2$, where $\rho > 1$ and $M_1, M_2 \in SL_2(\mathbb{R}^+)$ are like in Example 8. Then, clearly, $r_{\mathcal{A}} = \rho$ and, as explained within the same example, $s_{\mathcal{A}} = \log_\rho 2$, namely $r_{\mathcal{A}} = 2^{1/s_{\mathcal{A}}}$. Now, let $\mathcal{B}$ be the gasket generated by $\rho M_1, \sqrt{\rho}M_2$. Then $r_{\mathcal{B}} = r_{\mathcal{A}} = \rho$ but, this time, $s_{\mathcal{B}} = 2\log_\rho g$, where g is the golden ratio, so that $s_{\mathcal{B}} > s_{\mathcal{A}}$ and $2^{1/s_{\mathcal{B}}} = \rho^{\log_{g^2} 2} < r_{\mathcal{B}}$.*

Note that, since we are considering only finitely generated semigroups, the notions of *joint spectral radius* and *generalized spectral radius* coincide [Jun09].

3 Hausdorff dimension of attractors of finitely generated free semigroups of $PSL_2^{\pm}(\mathbb{R})$ and $PSL_2(\mathbb{C})$.

In this section, we show how the exponent of a free finitely generated semigroup $\boldsymbol{A} \subset SL_2^{\pm}(\mathbb{R})$ (resp. $\boldsymbol{A} \subset SL_2(\mathbb{C})$) is sometimes related to the Hausdorff dimension of the attractor of a generic orbit in $\mathbb{R}P^1$ (resp. $\mathbb{C}P^1$) of the semigroup $\Psi(\boldsymbol{A}) \subset PSL_2^{\pm}(\mathbb{R})$ (resp. $\Psi(\boldsymbol{A}) \subset PSL_2(\mathbb{C})$) naturally associated to $\boldsymbol{A}$ (equivalently, to the attractor of the IFS corresponding to $\Psi(\boldsymbol{A})$).

Let $\{f_1, \dots, f_m\}$ be a free set of linear automorphisms of $\mathbb{R}^2$ (resp. $\mathbb{C}^2$), preserving a volume 2-form *modulo sign*. With respect to any frame $\mathcal{E} = \{e_1, e_2\}$, these automorphisms are represented by matrices $A_i \in SL_2^{\pm}(\mathbb{R})$ (resp. $A_i \in SL_2(\mathbb{C})$). We denote by $\boldsymbol{A}$ the semigroup generated by the A_i and by $\psi_I \in PSL_2^{\pm}(\mathbb{R})$ (resp. $\psi_I \in PSL_2(\mathbb{C})$) the automorphism of $\mathbb{R}P^1 \simeq \mathbb{S}^1$ (resp. $\mathbb{C}P^1 \simeq \mathbb{S}^2$) naturally induced by f_I, $I \in \mathcal{I}^2$. The similarity between the characterization of the exponent $s_{\boldsymbol{A}}$ given in Theorem 2 and the formula for the Hausdorff dimension of a 1-dimensional IFS given in [Fal90] (Theorem 9.9, p. 126) suggests the following claim:

Theorem 4. *Assume that the f_i are all hyperbolic and that there exists some proper open set $V \subset \mathbb{R}P^1$ (resp. $V \subset \mathbb{C}P^1$) invariant under the ψ_i such that, for some affine chart $\varphi : \mathbb{R}P^1 \to \mathbb{R}$ (resp. some complex affine chart $\varphi : \mathbb{C}P^1 \to \mathbb{C}^1$), the ψ_i are contractions on $\varphi(\overline{V})$ with respect to the Euclidean distance, satsisfy $0 < a \leq |\psi_i'(v)| \leq c < 1$ for all $1 \leq i \leq m$, $v \in V$ and some constants a, c. Finally assume that the ψ_i satisfy the open set condition $\psi_1(V) \cap \psi_2(V) = \emptyset$. Let $R_{\boldsymbol{A}} = \cap_{k=1}^{\infty} \left(\cup_{|I|=k} \psi_I(V) \right)$ be the corresponding attractor. Then $2 \dim_H R_{\boldsymbol{A}} = s_{\boldsymbol{A}}$.*

Proof. Let us first consider the real case. In the chart φ by the hypothesis the ψ_i are a *cookie-cutter system* (see Chapter 4 of [Fal97]). In particular they satisfy the *principle of bounded distortion* (see Prop. 4.2 of [Fal97]), namely

$$|\psi_I(V)| \asymp |\psi_I'(x)|$$

for all $x \in \psi_I(V)$, where $|\psi_I(V)|$ is the diameter of $\psi_I(V)$. Let x_I be the stable fixed point of ψ_I and $\|\cdot\|_s$ the spectral norm. Then a direct calculation shows that $\psi_I'(x_I) = \|A_I\|_s^{-2}$ and, therefore,

$$|\psi_I(V)| \asymp \|A_I\|_s^{-2} \asymp \|A_I\|^{-2}.$$

Now, by Theorem 5.3 in Chapter 5 of [Fal97], we have also that $\dim_H R_{\boldsymbol{A}}$ is the unique solution to the equation

$$1 = \lim_{k\to\infty} \left[\sum_{|I|=k} |\psi_I(V)|^s \right]^{\frac{1}{k}}.$$

Since, on the other side,

$$\lim_{k\to\infty} \left[\sum_{|I|=k} |\psi_I(V)|^s \right]^{\frac{1}{k}} = \lim_{k\to\infty} \left[\sum_{|I|=k} \|A_I\|^{-2s} \right]^{\frac{1}{k}} = \xi_{\boldsymbol{A}}(2s),$$

it follows by Theorem 2 that $s_{\boldsymbol{A}} = 2\dim_H R_{\boldsymbol{A}}$.

In the complex case, the maps ψ_I are all conformal and so the same theorems mentioned above hold (see Section 5.5 in Chapter 5 of [Fal97]). $\square$

An interesting consequence of the previous theorem is the following constraint posed by geometry to the (algebraic) exponent of the semigroups satisfying its conditions:

Corollary 5. *Let $\boldsymbol{A} \subset SL_2^{\pm}(\mathbb{R})$ (resp. $\boldsymbol{A} \subset SL_2(\mathbb{C})$) be a semigroup satisfying the conditions of the theorem above. Then $s_{\boldsymbol{A}} \leq 2$ (resp. $s_{\boldsymbol{A}} \leq 4$).*

Proof. This is a direct consequence of the fact that the Hausdorff dimension of a subset of $\mathbb{R}^n$ cannot be bigger than n. $\square$

3.1 Matrices with non-negative entries

$SL_2^{\pm}(\mathbb{R}^+)$ is a source for several interesting semigroups that satisfy the hypotheses of Theorem 4. In this simple setting there is an elementary sufficient condition to determine whether a gasket is fast:

Proposition 10. *Consider a homomorphism $\mathcal{A} : \mathcal{I}^m \to SL_2^{\pm}(\mathbb{R}^+)$ and assume that all elements A_I, $|I| = 2$, have no entry equal to zero. Then $\mathcal{A}$ is fast.*

Proof. Let $A_{12} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ and $A_K = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, so that

$$A_{12K} = \begin{pmatrix} pa + qc & pb + qd \\ ra + sc & rb + bd \end{pmatrix}.$$

Assume, for the argument's sake, that $\|A_{12K}\| = ra + sc$. Then, since

$$pa + qc \geq \frac{\min\{p, q\}}{\max\{r, s\}}(ra + sc),$$

we have that, for every $M \in M_2(\mathbb{R}^+)$,

$$\|MA_{12K}\| \geq \|M\|(pa + qc) \geq \frac{\min\{p, q\}}{\max\{r, s\}}\|M\|\|A_{12K}\|.$$

By repeating this argument for every index of order 2 and denoting by c the smallest of these (finitely many) coefficients, we have that $\|MA_J\| \geq c\|M\|\|A_J\|$ for every $M \in M_2(\mathbb{R}^+)$ and $J \in \mathcal{J}^m$. In particular then, $\mathcal{A}$ is a fast homomorphism with coefficient not smaller than c. $\square$

Example 9. *Let $\mathcal{E} = \{e_1, e_2\}$ be a frame on $\mathbb{R}^2$ and $f_{1,2}$ defined by*

$$f_1(e_1) = e_1 + e_2, f_1(e_2) = e_2;\ f_2(e_1) = 2e_1 + e_2, f_2(e_2) = e_2.$$

With respect to $\mathcal{E}$, the f_i are represented by the matrices

$$F_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, F_2 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}.$$

The semigroup $\boldsymbol{F} = \langle F_1, F_2 \rangle \subset SL^-(\mathbb{N})$ is free because, if $F_I \in \boldsymbol{A}$, $I \in \mathcal{I}^2$, then the entries in F_I's lower row are equal to the entries in the upper row of the matrix $F_{I'}$ and, according to whether the upper left entry of F_I is larger or smaller than its lower left entry, we get either $I = 2I'$ or $I = 1I'$. Proceeding recursively this way, we see that there is no other index $J \neq I$ such that $F_J = F_I$. In particular then, $\boldsymbol{F}$ is a gasket. Moreover, $\boldsymbol{F}$ is hyperbolic: indeed, $\|F_I\| \geq \|F_1^{|I|}\|$, since F_2 has no entry smaller than the corresponding entry of F_1, and $\|F_1^k\| \simeq g^k$, where g is the golden ratio, because clearly $F_1^k = \begin{pmatrix} f_{k+2} & f_{k+1} \\ f_{k+1} & f_k \end{pmatrix}$, where f_k, $k \geq 1$, is the Fibonacci sequence $0, 1, 1, 2, 3, 5, \ldots$ *Finally, $\boldsymbol{F}$ is fast (with coefficient not smaller than $1/3$) by the previous proposition.*

The $\psi_{1,2}$ leave invariant the projective image of the positive cone $S(\mathcal{E})$ over $\mathcal{E}$, but they are not contractions on it. However, they are contractions over the projective image of the smaller cone $S(\mathcal{E}')$, with

$$\mathcal{E}' = \{e'_1 = (1 + \sqrt{3}, 2), e'_2 = (1 + \sqrt{3}, 1)\}.$$

Let $[x : y]$ be homogeneous coordinates on $\mathbb{R}P^1$ corresponding to $\mathcal{E}'$. In the canonical chart $\varphi = x/y$, the maps ψ_i induced by f_i can be written as

$$\psi_1(\varphi) = \frac{\varphi + 1}{\varphi},\ \psi_2(\varphi) = \frac{2\varphi + 1}{\varphi},$$

which reveals that this example coincides with Example 9.8 of [Fal90], coming from the theory of continued fractions.

To obtain analytical bounds for $s_{\boldsymbol{F}}$ we can use Theorem 2. Since both generators have an eigenvalue larger than 1, the norms of the terms $F_1F_2^k$ and $F_2F_1^k$ grow exponentially, so that we can get a good approximation of $\mu_{\boldsymbol{F},\ell}$ by truncating the sums after just a few terms. By considering only the terms with $k \leq 10$ in $\mu_{\boldsymbol{F},0}$ and solving the equation $\mu_{\boldsymbol{F},0}(s) = 2^s$ in this approximation, we get $s_{\boldsymbol{F}} \geq .51$, with a relative error of about 6% on the more precise estimate $s_{\boldsymbol{F}} \geq .54$ obtained by considering $k \geq 20$. Since $c = 1/3$, the first $\mu_{\boldsymbol{F},\ell}$ we can get upper bounds is $\mu_{\boldsymbol{F},3}$. Here we just mention that from $\mu_{\boldsymbol{F},8}$, considering the first 30 summands of all series that appear in its expression, we get $0.95 \leq s_{\boldsymbol{F}} \leq 1.76$. In terms of the dimension of $R_{\boldsymbol{F}}$, this translates to $0.474 \leq \dim_H R_{\boldsymbol{F}} \leq 0.877$. By evaluating $N_{\boldsymbol{F}}(k)$ for $k = 2^p$, $1 \leq p \leq 35$, we get the estimate $s_{\boldsymbol{F}} \simeq 1.062$ (see Table 1 for the corresponding values of $N_{\boldsymbol{F}}$), with a (heuristic) error of 2 on the last digit. This corresponds to the well-known fact $\dim_H R_{\boldsymbol{F}} \simeq 0.531$.

Example 10. *Consider now*

$$f_1(e_1) = e_1, f_1(e_2) = e_1 + e_2;\ f_2(e_1) = e_1 + e_2, f_2(e_2) = e_2.$$

The corresponding matrices (with respect to $\mathcal{E}$)

$$C_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, C_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

generate the semigroup $\boldsymbol{C} \subset SL_2(\mathbb{N})$ we already met in Examples 2 and 6. In particular, we already know that $\boldsymbol{C}$ is a parabolic fast gasket with coefficient $c = 1/2$. A direct check shows that the slowest and fastest growths, with respect to the order k of the multi-index I of $C_I \in \mathcal{A}$, correspond respectively to the pure powers C_i^k, for which $\|C_i^k\| = k$, and to the "cyclic" products $C_iC_{i+1}C_{i+2}\cdots C_{i+k-1}$, for which $\|C_iC_{i+1}C_{i+2}\cdots C_{i+k-1}\| \simeq g^k$, where the sums in the indices are meant "modulo 2" in the sense that 3 means 1, 4 *means 2 and so on. The reason why the golden ration g appears is that, similarly to the previous case,*

$$C_1C_2C_1\cdots C_{i+k-1} = \begin{pmatrix} f_{k+2} & f_{k+1} \\ f_{k+1} & f_k \end{pmatrix}$$

for k odd while if k is even the two rows get exchanged and analogously for the cyclic products beginning by C_2.

In the affine chart $\varphi : [x : y] \to x/(x+y)$, the maps ψ_i induced by the f_i can be written as

$$\psi_1(\varphi) = \frac{\varphi}{1+\varphi},\ \psi_2(\varphi) = \frac{1}{2-\varphi}$$

and the segment $S(\mathcal{E})$ maps into $[0,1]$. Note that this choice of chart corresponds to writing $e_1 = e'_1$ and $e_2 = e'_1 + e'_2$, expressing the f_i with respect to $\mathcal{E}' = \{e'_1, e'_2\}$ and using the canonical chart $y' = 1$ for the corresponding homogeneous coordinates $[x' : y']$. In terms of the semigroup, this corresponds to the adjunction via the matrix $M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. A direct calculation shows that the $|\psi'_i(\varphi)| \leq 1$ on $[0,1]$, with the equal sign holding at 0 for ψ_1 and at 1 for ψ_2, namely the IFS $\{\psi_1, \psi_2\}$ is parabolic.

Evaluating the Hausdorff dimension of the attractor $R_{\mathcal{C}_2}$ of a point $w \in (0,1)$ under the action of $\mathcal{C}_2$ is nevertheless an easy task. Indeed, since $\psi_1((0,1)) = (0,1/2)$ and $\psi_2((0,1)) = (1/2,1)$, the ψ_I, $|I| = k$, subdivide $(0,1)$ into 2^k disjoint segments $d_I = \psi_I(0,1)$ in such a way that $\cup_{|I|=k}\overline{d_I} = [0,1]$. Moreover, the length of these segments goes to zero for $k \to \infty$. Indeed, if $C_I = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{C}_2$, then $\psi_I(\varphi) = \frac{(a-c)\varphi + c}{(a+b-c-d)\varphi + c + d}$ and, therefore,

$$|d_I| = \left|\frac{c}{c+d} - \frac{a}{a+b}\right| = \frac{1}{(a+b)(c+d)},$$

from which we get

$$\frac{1}{4\|C_I\|^2} \leq |d_I| \leq \frac{1}{\|C_I\|}.$$

Hence, the orbit under the ψ_I of every $w \in (0,1)$ is dense in $(0,1)$ and, therefore, $\dim_H R_{\boldsymbol{C}} = 1$. Note that, since this IFS is not hyperbolic, Theorem 4 does not apply to it and therefore we cannot conclude that $s_{\boldsymbol{C}} = 2$.

Let us get analytical bounds for $s_{\boldsymbol{C}}$ via $\mu_{\boldsymbol{C},\ell}$. Unlike the previous example, the presence of parabolic elements in the semigroup does not allow to truncate *the series to just a few terms because of its very slow convergence. Since* $\|C_iC_{i+1}^k\| = k+1$, *we get easily that*

$$\mu_{\boldsymbol{C},0}(s) = \sum_{J\in\mathcal{J}^2} \|C_J\|^{-s} = 2\sum_{k=1}^{\infty} \|C_1C_2^k\|^{-s} = 2\sum_{k=2}^{\infty} k^{-s} = 2(\zeta(s)-1),$$

where $\zeta(s)$ *is the Riemann's zeta function. The solution of* $\mu_{\boldsymbol{C},0}(s) = 2^s$ *gives us the bound* $s_{\boldsymbol{C}} \geq 1.54$. *The first upper bound can be gotten from* $\mu_{\boldsymbol{C},2}$, *obtained by replacing the two terms of norm 2 in* $\mu_{\boldsymbol{C},0}(s)$, *namely* $\|C_{12}\|^{-s}$ *and* $\|C_{21}\|^{-s}$, *with, respectively,* $\mu_{\boldsymbol{C}A_{12}}(s)$ *and* $\mu_{\boldsymbol{C}A_{21}}(s)$. *A direct calculation shows that*

$$\mu_{\boldsymbol{C}A_{12}}(s) = \sum_{k=2}^{\infty}(2k+1)^{-s} + \sum_{k=4}^{\infty} k^{-s} = 2^{-s}\zeta(s,\frac{5}{2}) + \zeta(s) - 1 - 2^{-s} - 3^{-s},$$

where $\zeta(s,t)$ *is the Hurwitz zeta function, and, by symmetry, we know that* $\mu_{\boldsymbol{C}A_{12}} = \mu_{\boldsymbol{C}A_{21}}$. *Hence*

$$\mu_{\boldsymbol{C},2}(s) = 2\left(2\zeta(s) + 2^{-s}\zeta(s,\frac{5}{2}) - 2 - 2^{1-s} - 3^{-s}\right),$$

which gives the bounds $1.7 \leq s_{\boldsymbol{C}} \leq 3.93$ *as solutions of* $\mu_{\boldsymbol{C},2}(s) = 2^s$ *and* $\mu_{\boldsymbol{C},2}(s) = 2^{-s}$. *A numerical evaluation of* $N_{\boldsymbol{C}}(k)$ *for* $k = 2^p$, $1 \leq p \leq 21$, *(see Table 1 for the evaluated values) gives* $s_{\boldsymbol{C}} = 2.000002$ *with a (heuristic) error of 2 on the last digit. This and the evaluation of* $\dim_H R_{\boldsymbol{C}}$ *above strongly suggest that* $s_{\boldsymbol{C}} = 2$.

It is interesting to consider the following generalization of the previous example, namely the free semigroups $\boldsymbol{C}_{2,\alpha} \subset SL_2(\mathbb{R}^+)$ generated by

$$C_{1,\alpha} = \begin{pmatrix}\alpha & 0\\ 1/\alpha & 1/\alpha\end{pmatrix}, C_{2,\alpha} = \begin{pmatrix}1/\alpha & 1/\alpha\\ 0 & \alpha\end{pmatrix}.$$

In this case, in the same framework used above,

$$\psi_{1,\alpha}(\varphi) = \frac{\varphi}{\alpha^2 + (2-\alpha^2)\varphi},\ \psi_{2,\alpha}(\varphi) = \frac{1+(\alpha^2-1)\varphi}{2+(\alpha^2-2)\varphi}.$$

A direct check shows that, for every fixed $\alpha \in (1,2)$, the $\psi_{i,\alpha}$ are contractions on the invariant interval $[0,1]$ and that they satisfy the open set condition with respect to it. Let $R_{\boldsymbol{C}_{2,\alpha}}$ be the attractor of the orbit of any point $w \in (0,1)$ under the action induced by $\boldsymbol{C}_{2,\alpha}$. The very same argument used in the example above shows that $\dim_H R_{\boldsymbol{C}_{2,\alpha}} = 1$. As a corollary of Theorem 4 we get the following:

Proposition 11. *$s_{\boldsymbol{C}_{2,\alpha}} = 2$ for every $\alpha \in (1,2)$.*

Remark 4. *The restriction on the possible values of α looks more like an artificial effect of a poor choice for the distance function rather than a true property of the semigroups. We believe that by choosing an ad-hoc metric and maybe slightly modifying the argument the proposition above can be extended to the half-line $[1,\infty)$.*

3.2 Complex Sierpinsky Gaskets

Here we define a new class of complex self-projective sets whose construction is topologically equivalent to the one of the well-known Sierpinsky gasket.

Definition 5. *Let f_1, f_2, f_3 be volume-preserving linear automorphisms of $\mathbb{C}^2$ and denote by $A_1, A_2, A_3 \in SL_2(\mathbb{C})$ the corresponding matrices with respect to some coordinate system and by $\psi_1, \psi_2, \psi_3 \in PSL_2(\mathbb{C})$ their corresponding Möbius transformations acting on the Riemann sphere. Let $x_i \in \mathbb{C}P^1$ be the stable fixed point of ψ_i. We say that the semigroup $\boldsymbol{F}$ generated by the f_i (or, equivalently, the semigroup $\boldsymbol{A}$ generated by the A_i) is a* complex projective Sierpinski gasket *if $\psi_i(x_j) = \psi_j(x_i)$ for every pair $i \neq j$ and there exists a curvilinear triangle $T_{\boldsymbol{A}}$ having the x_i as vertices that is invariant under the action of the ψ_i.*

By construction, every such gasket $\boldsymbol{A}$ is free and satisfies the open set condition with respect to the interior of $T_{\boldsymbol{A}}$. Since the Möbius group $PSL_2(\mathbb{C})$ is transitive on triples of distinct points, we assume without loss of generality in the rest of this section that $T_{\boldsymbol{A}}$ has vertices $x_1 = [1:1]$, $x_2 = [i:1]$, $x_3 = [-1:1]$ with respect to homogeneous coordinates $[z:w]$ and use the affine chart $w = 1$ with complex coordinate $z = x + iy$ for all calculations.

Proposition 12. *Let f_1, f_2, f_3 be volume-preserving linear automorphisms with real spectrum having respectively $e_1 = (1,1)$, $e_2 = (i,1)$, $e_3 = (-1,1)$ as* *eigenvectors corresponding to the largest eigenvalue and assume that*

$$\begin{cases} \psi_1([e_3]) = \psi_3([e_1]) = u + iv, \\ \psi_2([e_1]) = \psi_1([e_2]) = is, \\ \psi_3([e_2]) = \psi_2([e_3]) = -u + iv. \end{cases}$$

A necessary condition for f_1, f_2, f_3 to generate a Sierpinski gasket symmetric with respect to the imaginary axes, namely such that $f_1(z) = \overline{f_2(-\overline{z})}$ and $f_3(z) = \overline{f_3(-\overline{z})}$, is that $\psi_1([e_3]) \in \Gamma$, where Γ is the circle

$$x^2 + y^2 - x(1 - s^2) - s^2 = 0. \tag{24}$$

For $s = 0$ the condition is sufficient for $u \in [1/5, \alpha]$, where $\alpha \simeq 0.651$.

Proof. A long but straightforward direct calculation shows that condition (24) is the only one coming from imposing that each one of the tetruples $[e_1]$, $[e_3]$, $\psi_1([e_3])$, $\psi_{11}([e_3])$ and $[e_1]$, $[e_2]$, $\psi_1([e_2])$, $\psi_{11}([e_2])$ identifies a single circumference. No further condition comes from ψ_3 and by symmetry we obtain an equivalent condition with respect to ψ_2.

When $s = 0$ another direct calculation shows that if $u < 1/5$ then e_1 is not anymore the eigenvector of f_1 corresponding to its largest eigenvalue. When $u = \alpha$ the circles Γ_{13} and Γ_{12} are tangent to each other and for $u > \alpha$ they intersect inside $T_{\boldsymbol{A}}$. $\square$

Example 11. *Let us give a short survey of the kind of geometry we meet in case of complex projective Sierpinski gaskets symmetric with respect to the imaginary axes. For $u = 16/25 \simeq \alpha$ we get the gasket*

$$A_1 = \frac{1}{\sqrt{544}} \begin{pmatrix} 20 & 12i \\ -3i & 29 \end{pmatrix}, A_2 = \frac{1}{\sqrt{24}} \begin{pmatrix} 4 & 4 \\ 1 & 7 \end{pmatrix}, A_3 = \frac{1}{\sqrt{24}} \begin{pmatrix} 4 & -4 \\ -1 & 7 \end{pmatrix}.$$

In Fig. 2 we show the orbit of a point under the action of the semigroup $\boldsymbol{A}_{\frac{16}{25}}$ generated by the A_i. The triangle $T_{\boldsymbol{A}_{\frac{16}{25}}}$ is convex and, correspondingly, the triangle $Z_{\boldsymbol{A}_{\frac{16}{25}}} = T_{\boldsymbol{A}_{\frac{16}{25}}} \setminus (\cup_{i=1}^3 T_{\boldsymbol{A}_{\frac{16}{25}} A_i})$ is concave. Each angle is almost zero because the sides of the triangle are almost tangent to each other, which corresponds to the fact that the limit value α is close to $16/25$. The restriction to $T_{\boldsymbol{A}_{\frac{16}{25}}}$ of corresponding maps ψ_i are contractive, so that Theorem 5 applies. A rough numerical evaluation of the exponent of $\boldsymbol{A}_{\frac{16}{25}}$ gives $s_{\boldsymbol{A}_{\frac{16}{25}}} \simeq 2.88$, so that $\dim R_{\boldsymbol{A}_{\frac{16}{25}}} \simeq 1.44$.

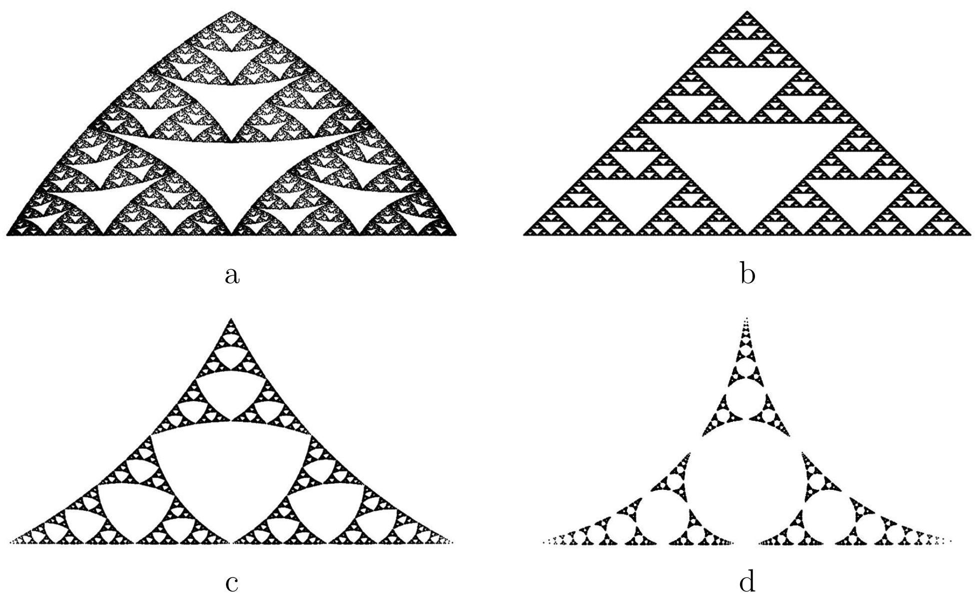

Figure 2: Attractors of complex self-projective Sierpinski gaskets: (a) $u = 16/25$, $s_{\boldsymbol{A}} \simeq 2.88$, $\dim R_{\boldsymbol{A}} \simeq 1.44$; (b) $u = 1/2$, $s_{\boldsymbol{A}} = 2\log_2 3$, $\dim R_{\boldsymbol{A}} = \log_2 3$; (c) $u = 9/25$, $s_{\boldsymbol{A}} \simeq 2.88$, $\dim R_{\boldsymbol{A}} \simeq 1.44$; (d) $u = 1/5$, $s_{\boldsymbol{A}} \simeq 2.60$, $\dim R_{\boldsymbol{A}} \simeq 1.30$. Above we show for each case the 19683 points of the orbit of a random point under the action of all matrices A_I of the gasket with $|I| = 9$.

By increasing u, the curvature of the sides increases (we consider negative the curvature of concave sides) until it gets zero for $u = 1/2$. The semigroup is now generated by

$$A_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ 0 & 2\end{pmatrix}, A_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}, A_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 0 & 2\end{pmatrix}.$$

In this case all sides are segments of straight lines and the gasket is diffeomorphic to the standard Sierpinski gasket in $\mathbb{R}^2$. It is easy to prove that $s_{\boldsymbol{A}_{\frac{1}{2}}} = 2\log_2 3$*. Correspondingly, we get the well-known result* $\dim R_{\boldsymbol{A}_{\frac{1}{2}}} = \log_2 3$.

By increasing u further, the curvature of the sides keeps increasing and, therefore, $T_{\boldsymbol{A}}$ becomes convex. For $u = 9/25$ the semigroup is generated by

$$A_1 = \frac{1}{45}\begin{pmatrix}3 & 6i\\ 2i & 11\end{pmatrix}, A_2 = \frac{1}{\sqrt{24}}\begin{pmatrix}3 & 3\\ -1 & 7\end{pmatrix}, A_3 = \frac{1}{\sqrt{24}}\begin{pmatrix}3 & -3\\ 1 & 7\end{pmatrix}.$$

The corresponding ψ_i are contractive over $T_{\boldsymbol{A}_{\frac{9}{25}}}$, so that Theorem 5 applies. A rough numerical evaluation of the exponent gives $s_{\boldsymbol{A}_{\frac{9}{25}}} \simeq 2.88$*, so that* $\dim R_{\boldsymbol{A}_{\frac{9}{25}}} \simeq 1.44$.

At the extremal value $u = 1/5$, every angle of the triangle is equal to π, namely every triangle $Z_{\boldsymbol{A}_{\frac{1}{5}}}$ is actually a circle. Indeed this gasket, generated by

$$A_1 = \begin{pmatrix}0 & i\\ i & 2\end{pmatrix}, A_2 = \frac{1}{2}\begin{pmatrix}1 & 1\\ -1 & 3\end{pmatrix}, A_3 = \frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & 3\end{pmatrix}.$$

is the well-known Apollonian gasket. This time, the corresponding ψ_i are only non-expansive, which corresponds to the fact that the generators of $\boldsymbol{A}_{\frac{1}{5}}$ are parabolic. Nevertheless, the numerical evaluation of the exponent gives $s_{\boldsymbol{A}_{\frac{1}{5}}} \simeq 2.60$*, compatible with the known estimate* $\dim R_{\boldsymbol{A}_{\frac{1}{5}}} = 1.30568...$ *(e.g. see [McM98]), and therefore suggests that the result of Theorem 4 holds even in the parabolic case.*

Finally, we point out that all these gaskets are fast. Here we outline the argument in case of the Apollonian gasket $\boldsymbol{A}_{\frac{1}{5}}$, but the same argument holds for all complex projective gaskets symmetric with respect to the imaginary axes. Note, first of all, that it is straightforward proving by induction that $\|A_I\| = |(A_I)_{22}|$ *for every matrix $A_I \in \boldsymbol{A}_{\frac{1}{5}}$. Now, consider the case*

$$A_I = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, A_{23} = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 5\end{pmatrix}, A_L = \begin{pmatrix}a & b\\ c & d\end{pmatrix}.$$

$N_{\boldsymbol{C}}(2^k)$ k=1,...,21	3, 15, 71, 287, 1231, 4911, 19831, 79279, 318383, 1273807, 5098247, 20391887, 81590055, 326364583, 1305483999, 5221928631, 20888160751, 83552534287, 334211194663, 1336844299831, 5347382007359
$N_{\boldsymbol{F}}(2^k)$ k=1,...,35	2, 7, 16, 34, 84, 151, 348, 679, 1546, 3034, 6546, 13476, 28409, 59578, 122139, 261698, 531191, 1144823, 2314772, 4986951, 10132768, 21667197, 44400099, 94074745, 194587388, 408651488, 852101402, 1777247239, 3726410796, 7738675037, 16274400897, 33739772516, 71002774691, 147235829060, 309533001058
$N_{\boldsymbol{A}_{\frac{1}{5}}}(1.4^k)$ k=1,...,29	3, 3, 5, 12, 29, 64, 123, 255, 594, 1372, 3222, 7388, 17636, 41302, 95748, 226607, 534766, 1271425, 3032945, 7221236, 17258732, 41170226, 98428382, 235630679, 563928974, 1351059074, 3238344644, 7765525872, 18627071753

Table 1: Several values of $N_{\boldsymbol{A}}$ for the *cubic semigroup* $\boldsymbol{C}$, the semigroup $\boldsymbol{F}$ of Example 9 and the Apollonian semigroup $\boldsymbol{A}_{\frac{1}{5}}$.

Then

$$A_{23L} = \frac{1}{2}\begin{pmatrix} a+c & b+d \\ a+5c & b+5d \end{pmatrix},$$

so that $\|A_{I23L}\| = \frac{1}{2}|\gamma(b+d)+\delta(b+5d)| \geq 2|\delta||d| \geq \frac{1}{3}\|A_I\|\|A_{23L}\|$. *The case of* A_{32} *is completely analogous to this. The remaining four combinations are instead analogous to the case of*

$$A_{12} = \frac{1}{2}\begin{pmatrix} -i & 3i \\ -2+i & 6+i \end{pmatrix}.$$

This time

$$A_{12L} = \frac{1}{2}\begin{pmatrix} -ia+3ic & -ib+3id \\ (-2+i)a+(6+i)c & (-2+i)b+(6+i)d \end{pmatrix}$$

and $\|A_{I12L}\| = \frac{1}{2}|\gamma(3id-ib)+\delta((i-2)b+(6+i)d)| \geq 2|\delta||d| \geq \frac{1}{5}\|A_I\|\|A_{12L}\|$. *Hence,* $\boldsymbol{A}_{\frac{1}{5}}$ *is a fast gasket with coefficient not smaller than* $1/5$.

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