A conjecture on the Hausdorff dimension of attractors of real self-projective Iterated Function Systems.

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Abstract

In a recent work [DeL14] we proved that, under natural conditions satisfied in several important examples, the rate of growth $1/s_A$ of norms of matrices in a semigroup $A \subset SL_2^{\pm}(\mathbb{R})$ (resp. $A \subset SL_2(\mathbb{C})$) dictates the Hausdorff dimension of the attractor R_A of the corresponding semigroups of projective transformations on $\mathbb{R}\mathrm{P}^1$ (resp. $\mathbb{C}\mathrm{P}^1$).

In the present work we start to study the higher dimensional case. In particular, we introduce some family of semigroups $A \subset SL_n(\mathbb{R})$ and we study numerically some concrete case for $n = 3$ and $n = 4$. Our results suggest that, for $n \geq 3$, $(n-1)s_A/n$ is a lower bound for the Hausdorff dimension of R_A.

1 Introduction

The present work is motivated by the unexpected numerical discovery of a non-trivial asymptotic property shared by two noteworthy gaskets (see below). This property led us to the definition of a critical exponent that, in case of gaskets invariant under the action of $PSL_2(\mathbb{R})$ or $PSL_2(\mathbb{C})$, completely determines the Hausdorff dimension of the gasket (see [DeL14]). Here we will provide numerical evidence on how such relation might extend to higher dimensional settings (see Conjecture 1).

Throughout the paper we will use the concepts of Hausdorff and box dimension of a set. We recall that the Hausdorff dimension of a subset F of a metric set (X, d) is the unique non-negative real number $\dim_H E$ such that

$$\dim_H E = \inf\{s \mid H^s(E) = 0\} = \sup\{s \mid H^s(E) = \infty\},$$

where

$$H^s(E) = \lim_{\delta\to 0} \inf\left\{\sum_{j=1}^{\infty} |U_j|^s \,\middle|\, E \subset \bigcup_{j=1}^{\infty} U_j, 0 < |U_j| < \delta\right\}$$

and $|U_j|$ is the diameter of U_j. Now, let $N_r(F)$ be the smallest number of sets of diameter r needed to cover F. Then the lower and upper box dimensions of F are defined, respectively, as

$$\underline{\dim}_B E = \liminf_{r\to 0} \frac{\log N_r(F)}{\log r}, \quad \overline{\dim}_B E = \limsup_{r\to 0} \frac{\log N_r(F)}{\log r}.$$

When $\underline{\dim}_B E = \overline{\dim}_B E = d$, we say that E has box-dimension $\dim_B E = d$. Recall that $\underline{\dim}_B E \geq \dim_H E$.

Throughout the paper we will use, for matrices, the norm $\|M\| = \max\{|M_{ij}|\}$ whenever not stated otherwise. Since all norms are equivalent in finite dimension, our results will not depend on this particular choice.

To clarify the main topic of the article, now we discuss in some detail two motivational examples.

1. The *Levitt-Yoccoz gasket* $\boldsymbol{C}_3$. Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be any frame of $\mathbb{R}^3$ and $[v] \in \mathbb{R}\mathrm{P}^2$ the direction corresponding to the non-zero vector $v \in \mathbb{R}^3$. The Levitt-Yoccoz gasket can be thought as the subset of $\mathbb{R}\mathrm{P}^2$ obtained by removing, from the triangle $T(\mathcal{E})$ with vertices $\{[e_1], [e_2], [e_3]\}$, the triangle with vertices $\{[e_1+e_2], [e_2+e_3], [e_3+e_1]\}$ and repeating this procedure recursively on the three triangles left (see Fig. 1). Of course by choosing different frames we get different sets, but all of them are projectively diffeomorphic to each other. Now, denote by $T_{k,\boldsymbol{C}_3} \subset T(\mathcal{E})$ the set obtained after repeating this procedure k times. Clearly $\boldsymbol{C}_3 = \cap_{k=1}^{\infty} T_{k,\boldsymbol{C}_3}$, i.e. we can get as close as we please to $\boldsymbol{C}_3$ with the sets $T_{k,\boldsymbol{C}_3}$ for large values of k. The set $\boldsymbol{C}_3$ can also be characterized as the (unique) subset of the triangle with vertices $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$ which is invariant under the action of the (free) subsemigroup of $PSL_3(\mathbb{N})$ generated by the projective automorphisms ψ_i, $i = 1, 2, 3$, induced by the following three $SL_3(\mathbb{N})$ matrices:

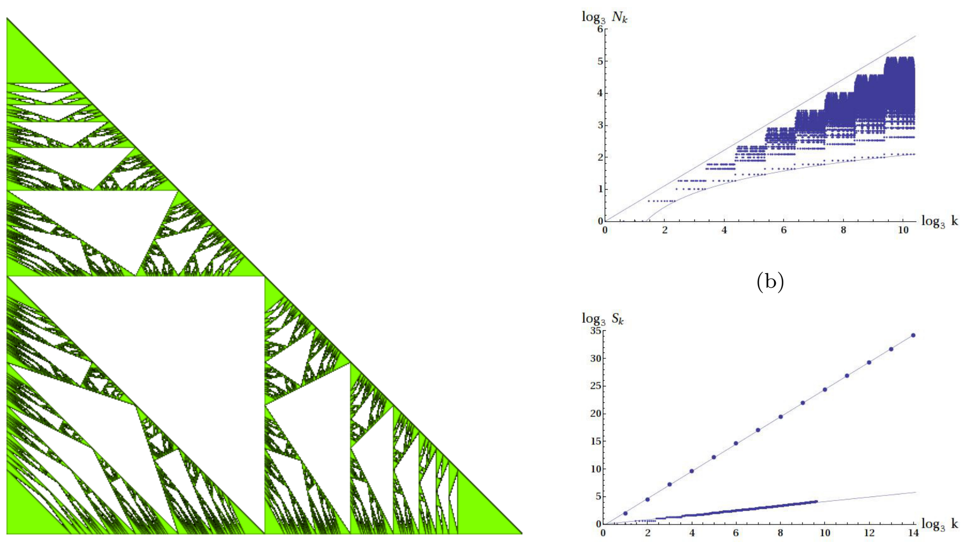
$$C_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \; C_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \; C_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

By abuse of notation we denote by $\boldsymbol{C}_3$ also the semigroup generated by these three matrices.

In the author's knowledge, the gasket $\boldsymbol{C}_3$ has been mentioned for the first time in 1991 in a paper by Arnoux and Rauzy [AR91], as an example in the context of interval exchange transformations. No property of the gasket has been investigated in that work, but it was conjectured that its Lebesgue measure is null. The gasket was then rediscovered in 1993 in a paper by G. Levitt [Lev93], as an example in the context of the dynamics of rotation pseudogroups. In that work it appears for the first time a picture of the fractal and a proof (by J.C. Yoccoz) of the conjecture above by Arnoux and Rauzy.

No further advances were made about $\boldsymbol{C}_3$ until it was independently rediscovered, in 2009, by I.A. Dynnikov and the present author [DD09] in the context of S.P. Novikov's theory of the magnetoresistance of metals under a strong magnetic field (e.g. see [Nov00, NM03]). From the mathematical point of view, the problem amounts to the dependance, on the Hamiltonian function H and a constant 1-form $B = B^k dp_k$, of the topology of the trajectories of the Poissonian dynamical system defined by H on the Poisson manifold $(\mathbb{T}^3, \{\}_B)$, where $\{p_i, p_j\}_B = \varepsilon_{ijk} B^k$ and ε_{ijk} is the totally antisymmetric Levi-Civita tensor. The closed 1-form B (magnetic field), although not exact, is the differential of the multi-valued function $B^i p_i$ and therefore is a multi-valued Casimir of the system [MR03]. According to a fundamental theorem by Dynnikov [Dyn92] this dynamical system is, in some sense, generically completely integrable. However, for some directions of the magnetic field it can be, loosely speaking, chaotic. In the generic case, this set of "chaotic directions" has Hausdorff dimension equal to 1 or less at every energy level. By a Novikov Conjecture [Nov00], this dimension is *strictly less* than 1, so that, presumably, the union of all of them for all levels is strictly less than 2. This property is important.

In [DD09], in case of a simple combinatorial model of Hamiltonian in which all chaotic directions are located at the same energy level, we were able to show that the set of chaotic directions is equal, modulo trivial smooth



(a) (c)

Figure 1: (a) The Cubic gasket $\boldsymbol{C}_3 \subset \mathbb{R}P^2$ in the triangle T with vertices $[0:0:1]$, $[1:0:1]$, $[0:1:1]$ (in homogeneous coordinates $[x:y:z]$). The picture shows (in green) the set $T_{7,\boldsymbol{C}_3}$ in the affine chart $z=1$. (b) Log-log plot of the norms N_k of the matrices $C_I \in \boldsymbol{C}_3$, $|I| \leq 11$, sorted in *lexicographic order*. The fastest growing norms are $\|C_i \cdot C_{i+1} \cdots C_{i+k}\| \simeq \alpha_3^k$, where sums of indices are intended "modulo 3" and $\alpha_3 \simeq 1.84$ is the Tribonacci constant. This corresponds, in the graph, to the line $N_k = k^{\log_3 \alpha_3}$, which bounds the norms from above. The slowest growing norms are the ones of the powers of the generators, $\|C_i^k\| = k$. These correspond, in the graph, to the line $N_k = \log_3 \frac{2k}{3}$, which bounds the norms from below. (c) Log-log plots of the functions $S_k = N_{\boldsymbol{C}_3}(k)$ (line with the largest slope), where $N_{\boldsymbol{C}_3}(k)$ is the number of elements of $\boldsymbol{C}_3$ whose norm is not larger than k, and $S_k = N_{\sigma(k)}$, where σ is any permutation of the indices sorting the N_k in *non-decreasing order* (line with the smallest slope). Only the first 40000 norms are included in the graph. Our numerical data suggests that $N_{\boldsymbol{C}_3}(k) \simeq Ak^s$ for $A \simeq 0.967$ and $s \simeq 2.444$ (the values of $N_{\boldsymbol{C}_3}(k)$ shown in the graph are *exact*, see Table 3). As expected, the norms sorted in non-decreasing order grow as $N_k \simeq k^{1/s}$. According to Conjecture 1, this entails that $\dim_H \boldsymbol{C}_3 \geq 1.63$.

transformations, to $\boldsymbol{C}_3$. It is therefore important to find bounds, in particular upper bounds, for its Hausdorff dimension. In [DD09] we provided an independent proof that $\boldsymbol{C}_3$ has zero Lebesgue measure, in order to show that the Hausdorff dimension of this gasket may indeed be smaller than 2. What makes hard finding non-trivial bounds for the Hausdorff dimension of $\boldsymbol{C}_3$ is that each projective automorphisms ψ_i induced by the matrices C_i has exactly one of the three vertices of the triangle $T(\mathcal{E})$ as fixed point and in that point it has Jacobian equal to $\mathbb{1}_3$, namely the iterated function system (IFS) $\{\psi_1, \psi_2, \psi_3\}$ is *parabolic* rather than *hyperbolic*[1]. Numerical results by the present author [DD09, DeL08] clearly indicate that such dimension is strictly smaller than 2. This fact was recently confirmed analytically by A. Avila, P. Hubert and S. Skripchenko [AHS13], but no stronger analytical bounds for $\dim_H \boldsymbol{C}_3$ are known to date.

We finally point out that, in 2013, Arnoux, showing no sign of awareness of the advances made on the subject since 1991, started studying *ex novo* the fractal in the context of Sturmian words in a joint work with S. Starosta [AS13] and provided a third, independent, proof that the measure of the fractal is null. They named the gasket "Rauzy gasket" but we believe that the gasket should rather be given the names of those who began studying its properties.

A key feature of the semigroup $\boldsymbol{C}_3$ is that its generators C_i, $i = 1, 2, 3$, have all eigenvalues equal to 1, so that the norms of their powers cannot grow faster than polynomially, while all their products C_iC_j, $i \neq j$, have an eigenvalue larger than 1, so that the norms of their powers grow exponentially. Hence, when sorted in lexicographic order[2], the norms N_i of the elements of $\boldsymbol{C}_3$ grow with quite different speeds, ranging from exponential to linear. The non-trivial asymptotic property of $\boldsymbol{C}_3$, which is our main motivation for the present work, is that the same norms appear instead to grow asymptotically as $N_{\sigma(i)} \simeq C\sigma(i)^r$, for some constants $C, r > 0$, when σ is any permutation of the indices which sorts the sequence $\{N_i\}$ in non-decreasing order. Equivalently, the number $N_{\boldsymbol{C}_3}(k)$ of elements of $\boldsymbol{C}_3$ whose norm is not larger than k grows as $N_{\boldsymbol{C}_3}(k) \simeq C'k^s$ for some $C' > 0$ and $s = 1/r$ (see Corollary 3 in [DeL14]). In Fig. 1 we show some numerical data relative to this case. We find that $s \simeq 2.44$ and, correspondingly, that $r \simeq 0.41$.

[1] Recall that a IFS $\{f_1, \dots, f_m\}$ on a metric space (M, d) is said *hyperbolic* when all f_i are contractions with respect to d and *parabolic* when all f_i are non-expanding maps.

[2] By this we mean that $N_1 = \|C_1\|$, $N_2 = \|C_2\|$, $N_3 = \|C_3\|$, $N_4 = \|C_1C_1\|$, $N_5 = \|C_1C_2\|$, ..., $N_{14} = \|C_1C_1C_1\|$ and so on.

Since the norm of the $\boldsymbol{C}_3$ matrices are related to the areas of the triangles (see Lemma 3), it has to be expected that the rate of their growth is related to the Hausdorff dimension of the fractal. According to Conjecture 1, indeed, the value found for s entails that $\dim_H \boldsymbol{C}_3 \geq 1.63$.

2. The *Apollonian gasket* $\boldsymbol{A}_3$. It was an enlightening seminar of Hee Oh about apollonian packings that brought to our attention the Apollonian gasket $\boldsymbol{A}_3 \subset \mathbb{C}P^1$. This complex self-projective fractal is possibly the fractal with the oldest ancestry, since its construction relies on a celebrated result of the Hellenistic mathematician Apollonius of Perga (ca 262 BC – ca 190 BC), known in his times as *The Great Geometer*. Apollonius' result, contained in the now–lost book *Tangencies* but fortunately reported by Pappus of Alexandria in his *Collection* [PapAD], published about five centuries later, concerns the existence of circles tangent to a given triple of objects that can be any combination of points, straight lines and circles. In particular, given three circles which are mutually externally tangent to each other (sometimes called the *four coins problem* [Old96]), there exist exactly two new circles tangent to all three, one externally and one internally (see Fig. 2). The three given circles plus any one of the new ones[3] form a *Descartes configuration*, since it was Descartes that stated the following remarkable relation between the curvatures $c_1, \dots, c_4$ of the four circles (see [Cox37] for details) memorialized three centuries later by the Chemistry Nobelist Frederick Soddy in his poem "The Kiss Precise" [Sod36] after rediscovering it independently: $2\sum_{i=1}^4 c_i^2 = \left(\sum_{i=1}^4 c_i\right)^2$.

Since Möbius transformations preserve circles and are transitive on triples of distinct points, they also act transitively on the set of all possible Descartes configurations; this fact suggests that their most natural environment is the Riemann sphere $\mathbb{C}P^1$ rather than the plane. Any Descartes configuration D divides $\mathbb{C}P^1$ in 4 curvilinear triangles T_i in such a way that every circle of D is one of the two Soddy circles of the remaining three circles of D. By drawing the new Soddy circle of each of the 4 triples we are left with 4 new Descartes configurations. By repeating this process recursively we generate an infinite osculating circle packing of $\mathbb{C}P^1$ which, not surprisingly, is called *Apollonian packing*.

Here we rather focus our attention on any one of the curvilinear triangles T and call *Apollonian gasket* $\boldsymbol{A}_3$ the set of points of T left after removing from

[3]In Soddy's honor the two new circles are called *Soddy's circles*.

T the interior of all Soddy circles inside it. Like in case of the cubic gasket, $\boldsymbol{A}_3$ can be characterized as the invariant set of a complex self-projective *parabolic* IFS. The fact that, thanks to the complex structure of $\mathbb{C}P^1$, $\boldsymbol{A}_3$ is *self-conformal* was exploited by Mauldin and Urbanski to prove some of its fundamental properties [MU98]. Unfortunately these techniques do not seem to extend to the previous (real) case, when the IFS maps are parabolic but not conformal.

In 1967 K.E. Hirst [Hir67] introduced the *Hirst semigroup* $\boldsymbol{H}$, namely the subsemigroup of $SL_4(\mathbb{N})$ generated by the matrices

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \; H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \; H_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix},$$

as an effective tool to generate the radii of the Soddy circles in the gasket.

In a series of fundamental contributions to the study of the Hausdorff dimension of the gasket [Boy70, Boy71, Boy72, Boy73a, Boy73b, Boy82], D.W. Boyd ultimately characterized this dimension in terms of the Hirst semigroup by proving (implicitly, in terms of the circles' curvatures) the following:

Theorem 1 (Boyd)**.** *Let $N_{\boldsymbol{H}}(k)$ be the number of matrices of the semigroup $\mathcal{H}$ with norm*[4] *not larger than k. Then:*

1. $\lim_{k\to\infty} \frac{\log N_{\boldsymbol{H}}(k)}{\log k} = s < \infty$;
2. $\dim_H \boldsymbol{A}_3 = s \simeq 1.306$.

Note that, in the meantime, both results have been improved. Recently, Kontorovich and Oh [KO11] strengthened point 1 by proving that there are constants $A, B > 0$ such that $Ak^s \leq N_{\boldsymbol{H}}(k) \leq Bk^s$ for almost all k. About point 2, the most precise evaluation of $\dim_H \boldsymbol{A}_3$ to date is, instead, due to McMullen [McM98] that, using the Apollonian *group*, namely the Schottky group whose residual set is the Apollonian gasket, found that $\dim_H \boldsymbol{A}_3 \simeq 1.3056688$ (see also Section 3.3).

A first similarity between the Apollonian and the Levitt-Yoccoz gasket comes from the fact that the distribution of the norms of the matrices in

[4]Since all norms are equivalent in finite dimension, this is true for any norm.

$\boldsymbol{H}$ is qualitatively the same as in the semigroup $\boldsymbol{C}_3$ (see Fig. 1 and Fig. 2). Note that, unlike in $\boldsymbol{C}_3$, only one of the generators of $\boldsymbol{H}$, namely H_1, has all eigenvalues equal to 1. The fastest growth in norms then, in this case, comes from the powers of the matrix H_3, which has the highest eigenvalue $\lambda \simeq 2.89$ among the three generators. This correponds, in the graph in Fig. 2(c), to the line $N_k = k^{\log_3 \lambda}$, which bounds the norms from above. The slowest growth comes instead from the powers of H_1, namely $\|H_1^k\| = k^2$, $k > 1$. This correponds, in the graph, to the line $N_k = (\log_3(2k-1)-1)^2$, which bounds the norms from below. As shown by Boyd though, even in this case, if σ is a permutation of the indices that sorts the norms in non-decreasing order, then $N_{\sigma(k)} = C\sigma(k)^r$, where $r = 1/s \simeq 0.766$, and, equivalently, the number of elements with norm not larger than k grows as $N_{\boldsymbol{H}}(k) \simeq Ck^s$ (see Fig. 2(d)).

A second similarity comes from the fact that $\boldsymbol{A}_3$, as shown in [DeL12, DeL14], can be seen as the invariant set of the *parabolic* Kleinian IFS corresponding to the subsemigroup of $SL_2(\mathbb{C})$ generated by the matrices

$$A_1 = \begin{pmatrix} 0 & i \\ i & 2 \end{pmatrix},\ A_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix},\ A_3 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix},$$

which by abuse of notation we will denote too by $\boldsymbol{A}_3$. Just like in case of $\boldsymbol{C}_3$, each generator of $\boldsymbol{A}_3$ has both eigenvalues equal to 1, so that the norms of their powers grow linearly, but each mixed product has an eigenvalue with modulus larger than 1, so that their powers grow exponentially. The asymptotics of the norms of the elements of $\boldsymbol{A}_3$ looks numerically the same as in $\boldsymbol{C}_3$. In Example 11 of [DeL14] we showed that $\lim_{k\to\infty} \log N_{\boldsymbol{A}_3}(k)/\log k$ converges to a finite limit s and conjectured that $s = 2\dim_H \boldsymbol{A}_3$, based on numerical evidence (see Table 3) and on the fact that such relation holds for similar semigroups that induce *hyperbolic* IFSs on $\mathbb{C}\mathrm{P}^2$.

It is interesting now to compare the two previous non-trivial gaskets with a simpler one, the *Sierpinski gasket* $\boldsymbol{S}_3 \subset \mathbb{R}^2$. This self-affine set is less ancient than the Apollonian one, having been introduced in the Mathematics literature by W. Sierpinski only in 1915 [Sie15], but it does have nevertheless a long history too since its pattern has been known and used in art for about a millennium [PA02] (see Fig. 3). Its dimension is easily calculated: $\dim_H \boldsymbol{S}_3 = \log_2 3$ (e.g. see [Fal90]). Since both $PSL_3(\mathbb{R})$ and $PSL_2(\mathbb{C})$ contain a subgroup isomorphic to the group of affine transformations of the plane, the Sierpinski gasket can also be seen as a real (respectively complex)

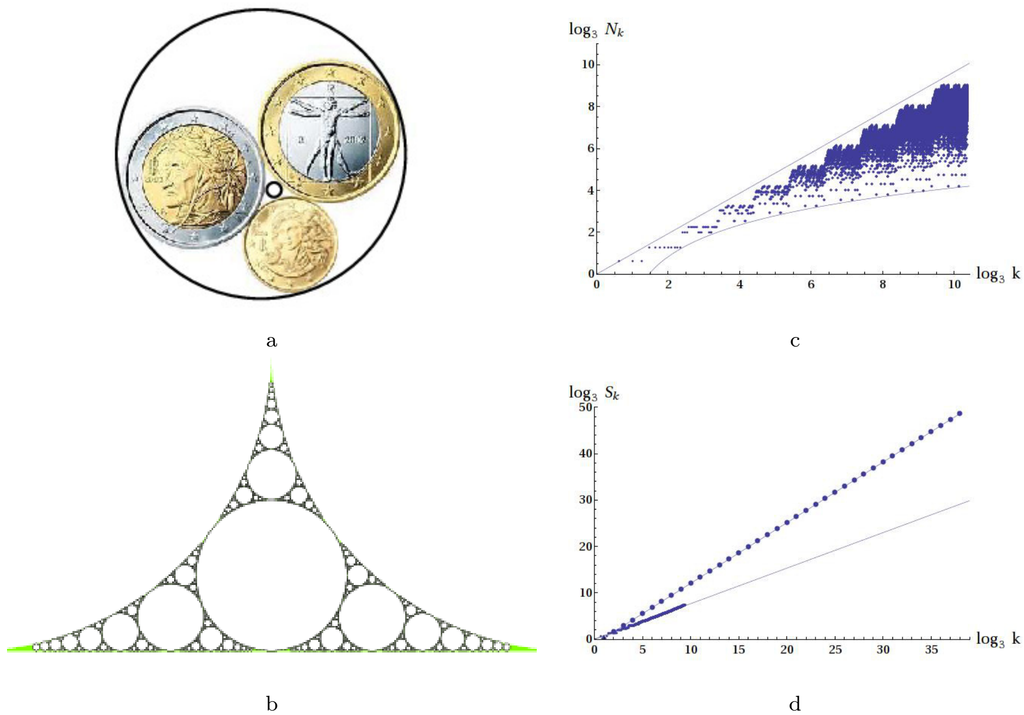


Figure 2: (a) Inscribed and circumscribed circles in the *four coins problem*. (b) Apollonian gasket $\boldsymbol{A}_3$ in the curvilinear triangle with vertices at $[1:1]$, $[-1:1]$, $[i,1]$ (in homogeneous coordinates $[z:w]$) and mutually tangent arcs of cirles as sides, represented in the affine chart $w=1$. The picture shows (in green) the set $T_{7,\boldsymbol{A}_3}$. (c) Log-log plot of the norms N_k of the matrices of the semigroup $\boldsymbol{H}$ sorted in lexicographic order. The fastest growing norms are $\|H_3^k\| \simeq \lambda^k$, where $\lambda \simeq 2.89$ is the largest modulus eigenvalue of H_3. This corresponds to the line $N_k = k^{\log_3 \lambda}$, which bounds the norms from the above. The slowest growing norms are $\|H_1^k\| = k^2$, $k > 1$. This corresponds to the line $N_k = (\log_3(2k-1)-1)^2$, which bounds the norms from below. (d) Log-log plots of the function $S_k = N_{\boldsymbol{H}}(k)$ (line with the largest slope), where $N_{\boldsymbol{H}}(k)$ is the number of elements of $\boldsymbol{H}$ with norm not larger than k, and $S_k = N_{\sigma(k)}$ (line with the smallest slope), where σ is any permutation of the indices sorting the N_k in *non-decreasing order*. The points shown in the graph for $N_{\boldsymbol{H}}(k)$ are exact, see Table 3. Only the first 40000 norms are included in the graph of $N_{\sigma(k)}$.

Figure 3: (left) Image of the Sierpinski gasket in the triangle T with vertices $(0,0)$, $(1,0)$, $(0,1)$. In the picture it is shown, in green, the set $T_{7,\boldsymbol{S}_3}$. (right) Detail of a cosmatesque [PA02] mosaic dated about 11th–12th century (photo taken by the author at the Phillips Museum in Washington, DC).

self-projective fractal of $\mathbb{R}P^2$ (respectively $\mathbb{C}P^1$).

A semigroup having the Sierpinski fractal as its attractor set in $\mathbb{R}\mathrm{P}^2$ is, for example, the one generated by the matrices

$$S_1^{\mathbb{R}} = \frac{1}{\sqrt[3]{2}} \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \; S_2^{\mathbb{R}} = \frac{1}{\sqrt[3]{2}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \; S_3^{\mathbb{R}} = \frac{1}{\sqrt[3]{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

In $\mathbb{C}\mathrm{P}^1$, in turn, we can use the semigroup induced by the matrices

$$S_1^{\mathbb{C}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 0 & 2 \end{pmatrix}, \; S_2^{\mathbb{C}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \; S_3^{\mathbb{C}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}.$$

The regularity of the matrices $S_i^{\mathbb{R}}$ and $S_i^{\mathbb{C}}$ makes possible to perform simple direct calculations that illustrate the main points of this paper.

Consider first the real version. Let $\|S\|_\infty$ be the norm given by the maximum absolute row sum of S. Since all lines of the $S_i^{\mathbb{R}}$ sum to 2 then $\|S_{i_1}^{\mathbb{R}} \cdots S_{i_p}^{\mathbb{R}}\|_\infty = (2^{-1/3})^p \cdot 2^p = 2^{2p/3}$ for every $p \geq 1$. Hence in the sphere of radius k lie

$$N(k) = \sum_{p=0}^{\lfloor \frac{3}{2}\log_2 k \rfloor} 3^p = \frac{3^{\lfloor \frac{3}{2}\log_2 k+1 \rfloor} - 1}{2}$$

products of the $S_i^{\mathbb{R}}$, where $\lfloor 3\log_2 k/2 \rfloor$ is the integer part of $3\log_2 k/2$, and therefore

$$s_{\mathbb{R}} = \lim_{k\to\infty} \frac{\log_2 N(k)}{\log_2 k} = \frac{3}{2}\log_2 3.$$

Note that $s_{\mathbb{R}}$ is also equal to the exponent that separates the values s for which the series $\sum_{S\in\langle S_i^{\mathbb{R}}\rangle} \|S\|^{-s}$ diverges from those for which it converges, where the sum is extended to all elements of the semigroup freely generated by the $S_i^{\mathbb{R}}$. Finally, note that the following relation holds between the Hausdorff dimension of the Sierpinski gasket and the rate growth: $3\dim_H \boldsymbol{S}_3 = 2s_{\mathbb{R}}$.

Consider now the complex version. Endow $M_2(\mathbb{C})$ with the norm $\|S\|$ given by the largest modulus of the entries of S. Since the last row of each $S_i^{\mathbb{C}}$ is $(0,2)$, then $\|S_{i_1}^{\mathbb{C}}\cdots S_{i_p}^{\mathbb{C}}\| = (2^{-1/2})^p \cdot 2^p = 2^{p/2}$ for every $p \geq 1$. Hence in this case

$$N(k) = \sum_{p=0}^{\lfloor 2\log_2 k\rfloor} 3^p = \frac{3^{\lfloor 2\log_2 k+1\rfloor} - 1}{2}$$

and therefore

$$s_{\mathbb{C}} = \lim_{k\to\infty} \frac{\log_2 N(k)}{\log_2 k} = 2\log_2 3.$$

Similarly to what happens in the real case, $s_{\mathbb{C}}$ is also equal to the exponent that separates the values s for which the series $\sum_{S\in\langle S_i^{\mathbb{C}}\rangle} \|S\|^{-s}$ diverges from those for which it converges, where the sum is extended to all elements of the semigroup freely generated by the $S_i^{\mathbb{C}}$. Note that in this case the relation between the Hausdorff dimension of the Sierpinski gasket and the norms' growth rate is the following: $2\dim_H \boldsymbol{S}_3 = s_{\mathbb{C}}$.

For thorough surveys on the Sierpinski gasket and, especially, on the more challenging Apollonian gasket, we refer the reader to the book by A.A. Kirillov [Kir13], the series of papers by Lagarias, Mallows, Wilks and Yan [GLM+03, GLM+05, GLM+06] and the recent article by Sarnak [Sar11].

The examples above suggest two interesting research directions.

On one side, the norms' asymptotics in several semigroups $\mathcal{A}$ of linear transformations on $\mathbb{R}^n$ leads to the association to them of a critical exponent $s_{\mathcal{A}}$. A natural condition for the existence and finiteness of $s_{\mathcal{A}}$ was studied in [DeL14]. In Section 2 we will generalize the Levitt-Yoccoz semigroup $\boldsymbol{C}_3$ by defining the concept of *real projective Sierpinski Gaskets* (in short, *Sierpinski Gasket*) and use such condition to prove analytically the numerical observations above on the aymptotics of those gaskets. As a byproduct, we will moreover show that $s_{\mathcal{A}}$ determines also the asymptotics of crucial geometrical parameters of the attractors $R_{\mathcal{A}}$ of the projective semigroups induced by $\mathcal{A}$ on $\mathbb{R}\mathrm{P}^{n-1}$.

On the other side, the relation between $s_{\mathcal{A}}$ and the geometry of $R_{\mathcal{A}}$ suggests that $s_{\mathcal{A}}$ might be somehow related to the Hausdorff dimension of $R_{\mathcal{A}}$. In [DeL14], Theorem 4, we proved that, under some natural conditions, in case of 2×2 real or complex matrices the exponent $s_{\mathcal{A}}$ actually determines completely $\dim_H R_{\mathcal{A}}$. In higher dimension it is to be expected a weaker relation. In Section 3 we present and discuss our numerical results on this matter for several significant examples of Sierpinski Gaskets coming from semigroups of 3×3 and 4×4 matrices and, at the end of the section, we briefly discuss also the case of the Apollonian Gasket.

The main result of this work, based on the numerical and analytical results on the box dimension of self-affine and self-projective fractals presented in Section 3, is the following:

Conjecture 1. *Let $\boldsymbol{A} \subset SL^{\pm}_n(\mathbb{R})$, $n \geq 3$, be a real projective Sierpinski gasket with exponent $s_{\boldsymbol{A}}$ and let $R_{\boldsymbol{A}}$ be its attractor. Then, under suitable natural assumptions, $n \dim_H R_{\boldsymbol{A}} \geq (n-1)s_{\boldsymbol{A}}$.*

Note that, since the box dimension of a set is always not smaller than its Hausdorff dimension, the conjecture entails the same relation for $\dim_B R_{\boldsymbol{A}}$.

2 Geometry of the Sierpinski Gaskets

The problem of determining the Hausdorff dimension of self-projective attractors of subsemigroups of $PSL_n(\mathbb{R})$, $n > 2$, is non-trivial; in fact, it is well known that even the simpler subcase of self-affine attractors is far from trivial (e.g. see [Fal88, FL98, ABVW10, FM11]). Because of this, and in order to provide motivation for the interest of real self-projective sets, we restrict our attention to the following particular case:

Definition 1. *Let $\boldsymbol{F} = \langle f_1, \dots, f_n \rangle$ be a free semigroup of volume-preserving linear automorphisms of $\mathbb{R}^n$ and $\psi_1, \dots, \psi_n \in PSL_n(\mathbb{R})$ the induced projective automorphisms of $\mathbb{R}P^{n-1}$. Let $\mathcal{E} = \{e_1, \dots, e_n\}$ be a n-frame of $\mathbb{R}^n$ and $\mathcal{E}^* = \{\varepsilon_1, \dots, \varepsilon_n\}$ its dual frame. We say that $\boldsymbol{F}$ is a* real projective Sierpinski gasket *over $\mathcal{E}$ if the following conditions are satisfied:*

1. $f_i = A^k_{ij} e_k \otimes \varepsilon^j$ *with* $A^k_{ij} \geq 0$;

2. $f_i(e_i) = \lambda_i e_i$, *with* $\lambda_i = \max_{1\leq j\leq n}\{A^j_{ij}\}$;

3. $f_i(e_j) = \alpha e_i + \beta e_j$ *with* $\alpha, \beta > 0$ *for* $i \neq j$;

4. $\psi_i([e_j]) = \psi_j([e_i])$, $i \neq j$.

We say that $\mathcal{E}$ *is a* proper frame *for* $\boldsymbol{F}$. *More generally, given* $m < n$ *of the* f_i*, we say that they are a Sierpinski gasket if there exist automorphisms* $f_{m+1}, \ldots, f_n$ *such that* $\langle f_1, \ldots, f_n \rangle$ *is a Sierpinski gasket.*

Note that conditions 1–3 above imply that *span*$\{e_i\}$ is the only eigenspace of f_i corresponding to its largest eigenvector, so that every proper frame for $\boldsymbol{F}$ identifies the same n points $[e_i]$ on $\mathbb{RP}^{n-1}$.

Denote by $C(\mathcal{E})$ the *positive cone* over $\mathcal{E}$, namely the convex hull of the set $\cup_{i=1}^n\{\lambda e_i, \lambda > 0\}$. Then its projection on $\mathbb{RP}^{n-1}$ is the same for every proper frame of $\boldsymbol{F}$ and we denote it by $T_{\boldsymbol{F}}$. This set is a $(n-1)$-simplex with the n points $[e_i]$ as vertices. By points 2 and 3 of the definition above, $[e_i]$ is a fixed point for ψ_i and each set $T_{\boldsymbol{F} f_i} \stackrel{\text{def}}{=} \psi_i(T_{\boldsymbol{F}})$ is a $(n-1)$-simplex having in common with every other $T_{\boldsymbol{F} f_j}$, $i \neq j$, the vertex $\psi_i([e_j])$. Like in case of the $(n-1)$-dimensional standard Sierpinski gasket in $\mathbb{R}^{n-1}$, the difference between $T_{\boldsymbol{F}}$ and $\cup_{i=1}^n T_{\boldsymbol{F} f_i}$ is the interior of a convex polyhedron with $n(n-1)/2$ vertices that we denote by $Z_{\boldsymbol{F}}$.

By repeating this procedure recursively we see that, at every step $k > 0$,

$$T_{k,\boldsymbol{F}} \stackrel{\text{def}}{=} \bigcup_{|I|=k} T_{\boldsymbol{F} f_I} = T_{\boldsymbol{F}} \setminus \left[\bigcup_{|I|<k} Z_{\boldsymbol{F} f_I} \right]$$

It is standard to call $R_{\boldsymbol{F}} = \cap_{k\geq 0} T_{k,\boldsymbol{F}}$ the *attractor* of $\boldsymbol{F}$.

For sake of simplicity and conciseness we limit our discussion to the following subclass of Sierpinski gaskets:

Definition 2. *We say that a Sierpinski gasket* $\boldsymbol{F} = \langle f_1, \ldots, f_m \rangle$ *is* simple of the first kind *when each* f_i *has only one distinct eigenvalue (therefore equal to 1) and* simple of the second kind *when it has exactly two distinct eigenvalues and the eigenspace corresponding to the larger one is 1-dimensional.*

Example 1. *The most important 1-parameter family of simple Sierpinski gaskets we discuss in this paper is the one of* cubic *semigroups* $\boldsymbol{C}_n^\alpha = \{f_1^\alpha, \dots, f_n^\alpha\}$, $\alpha \geq 1$,

$$f_i^\alpha(e_j) = \alpha^{-\frac{1}{3}} \begin{cases} \alpha e_i, & i = j, \\ e_i + e_j, & i \neq j. \end{cases}$$

These simple gaskets are all of the second kind except for $\alpha = 1$*. For* $n = 3$ *the* f_i^α *are represented, with respect to any proper frame, by the matrices*

$$A_1^{\alpha,3} = \alpha^{-\frac{1}{3}} \begin{pmatrix} \alpha & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_2^{\alpha,3} = \alpha^{-\frac{1}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 1 & \alpha & 1 \\ 0 & 0 & 1 \end{pmatrix}, A_3^{\alpha,3} = \alpha^{-\frac{1}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & \alpha \end{pmatrix}.$$

We call them cubic *because, for* $\alpha = 1$*, we get the Levitt-Yoccoz gasket* $\boldsymbol{C}_3 \stackrel{\text{def}}{=} \boldsymbol{C}_3^1$*, which is related to the regular* cubic *skew polyhedron* $\{4, 6|4\}$ *(see [DD09] for details). Note also that, as already shown in the introduction,* $\boldsymbol{C}_3^2$ *is the (real projective generalization of the) standard Sierpinski gasket* $\boldsymbol{S}_3$*. See Fig. 1 and Fig. 6 for two examples of (approximations of) residual sets of a cubic gasket with* $\alpha = 1$*.*

Our main goal for this section is to explain the asymptotic behaviour observed in case of the semigroup $\boldsymbol{C}_3$, shown in Figure 1. In order to achieve this, we will prove that the function $\log N_{\boldsymbol{F}}(r)/\log r$ converges to a finite limit for $r \to \infty$ for every simple real projective Sierpinski gasket.

Definition 3. *We denote by* $\mathcal{I}^m$ *the set of multi-indices in* m *variables and by* $\mathcal{J}^m$ *the set of* next-to-diagonal *multi-indices* $J = i_1 i_2 \dots i_\ell$*, namely such that* $i_1 \neq i_2 = \dots = i_\ell$*.*

We say that a free semigroup $\boldsymbol{F} = \langle f_1, \dots f_m \rangle$ *is* fast *if there is a constant* $c > 0$ *such that* $\|f_{IJK}\| \geq c\|f_I\|\|f_{JK}\|$ *for every* $I, K \in \mathcal{I}^m$ *and* $J \in \mathcal{J}^m$*.*

Roughly speaking, a gasket is fast if the only elements f_I for which the norm grows "slowly" are those where $I = ii \dots i$, namely only the powers of the generators. The reason for introducing this concept is the following result:

Theorem 2 ([DeL14])**.** *Let* $\boldsymbol{F}$ *be a free finitely generated fast semigroup and let* $N_{\boldsymbol{F}}(r)$ *the number of elements of* $\boldsymbol{F}$ *with norm not larger than* r*. Then*

$$\lim_{r \to \infty} \frac{\log N_{\boldsymbol{F}}(r)}{\log r} = s_{\boldsymbol{F}} < \infty.$$

Hence, to prove the asymptotic behaviour of norms in $\boldsymbol{C}_3$ (and any other simple Sierpinski gasket) it is enough to show that every such gasket is fast. We need first to prove the following two technical lemmas. Consider the dual semigroup $\boldsymbol{F}^* \subset Aut((\mathbb{R}^n)^*)$ of a Sierpinski gasket.

Lemma 1. *Let $\boldsymbol{F}$ be a simple Sierpinski gasket over a n-frame $\mathcal{E} = \{e_1, \dots, e_n\}$ of $\mathbb{R}^n$ generated by maps*

$$f_i(e_i) = \alpha_i e_i,\ f_i(e_j) = \beta_{ij} e_i + \gamma_i e_i.$$

Then $\boldsymbol{F}^*$ is a simple Sierpinski gasket over the frame $\mathcal{H} = \{\eta^1, \dots, \eta^n\}$ of $(\mathbb{R}^n)^*$ defined by

$$\eta^i = \sum_{j \neq i} \beta_{ij} \varepsilon^j + (\alpha_i - \gamma_i)\varepsilon^i. \tag{1}$$

Proof. By direct calculation we see that

$$f_i^*(\eta^i) = \sum_{j \neq i} \beta_{ij} f_i^*(\varepsilon^j) + (\alpha_i - \gamma_i) f_i^*(\varepsilon^i) =$$

$$= \sum_{j \neq i} \beta_{ij} \varepsilon^j + (\alpha_i - \gamma_i)(\alpha_i \varepsilon^i + \sum_{j \neq i} \beta_{ij} \varepsilon^j) =$$

$$= \alpha_i \left(\sum_{j \neq i} \beta_{ij} \varepsilon^j + (\alpha_i - \gamma_i)\varepsilon^i \right) = \alpha_i \eta^i$$

and

$$f_i^*(\eta^k) = \sum_{j \neq k} \beta_{kj} f_i^*(\varepsilon^j) + (\alpha_k - \gamma_k) f_i^*(\varepsilon^k) =$$

$$= \sum_{j \neq k,i} \beta_{kj} \varepsilon^j + \beta_{ki} \left(\sum_{j \neq i} \beta_{ij} \varepsilon^j + \alpha_i \varepsilon^i \right) + (\alpha_k - \gamma_k)\varepsilon^k = \beta_{ki} \eta^i + \gamma_i \eta^k.$$

□

Lemma 2. *Let $\boldsymbol{F} = \langle f_i \rangle \subset Aut(V^n)$ be a simple Sierpinski gasket over $\mathcal{E} = \{e_i\}$. Then the following inequalities hold:*

$$\|f_I\|_{\ell^1} \leq C \min_{\substack{1 \leq k,k' \leq n \\ k \neq k'}} \{\|f_I(e_k)\|_{\ell^1} + \|f_I(e_{k'})\|_{\ell^1}\}, \qquad \text{if } \boldsymbol{F} \text{ is of the first kind.}$$

$$\|f_I\|_{\ell^1} \leq C \min_{1 \leq k \leq n} \|f_I(e_k)\|_{\ell^1}, \qquad \text{if } \boldsymbol{F} \text{ is of the second kind,} \tag{2}$$

for some $C > 0$, where $\|\omega\|_{\ell^1} = \sum_{1 \leq j \leq n} |\omega_j|$.

Proof. Let $\mathcal{H} = \{\eta^i = \sum \hat{\beta}^i_j \varepsilon^j\}$ be the proper frame for $\boldsymbol{F}^*$ introduced in (1), where $\hat{\beta}^i_j = \beta_{ij} > 0$, $j \neq i$, and $\hat{\beta}^i_i = \alpha_i - \gamma_i \geq 0$. Clearly $\omega = \sum_{1\leq i\leq n} \eta^i \in C(\mathcal{H})$. By the previous proposition, $\omega_I = f_I^*(\omega) \in C(\mathcal{H})$ for all $I \in \mathcal{I}^n$. This means that $\omega_I = \sum_{1\leq i\leq n} (\omega_I)_i \varepsilon^i = \sum_{1\leq i\leq n} \lambda_i \eta^i$ (with $\lambda_i \geq 0$ and $\sum_{1\leq i\leq n} \lambda_i > 0$) so that $(\omega_I)_i = \sum_{1\leq j\leq n} \lambda_j \hat{\beta}^j_i$ and therefore

$$\|\omega_I\|_{\ell^1} = \sum_{1\leq i\leq n} (\omega_I)_i = \sum_{1\leq i,j\leq n} \lambda_j \hat{\beta}^j_i \leq n \max_{1\leq i,j\leq n} \{\hat{\beta}^i_j\} \sum_{1\leq i\leq n} \lambda_i.$$

Now note that $(\omega_I)_k \geq (\min_{\hat{\beta}^i_k > 0} \hat{\beta}^i_k) \sum_{\hat{\beta}^i_k > 0} \lambda_i$ is always a non-empty condition.

If $\boldsymbol{F}$ is of the second kind then $\hat{\beta}^i_j > 0$ for all i, j, so that

$$\|\omega_I\| \leq n \frac{\max\limits_{1\leq i,j\leq n} \{\hat{\beta}^i_j\}}{\min\limits_{1\leq i\leq n} \{\hat{\beta}^i_k\}} (\omega_I)_k$$

for all $1 \leq k \leq n$.

If $\boldsymbol{F}$ is of the first kind then $\hat{\beta}^i_k = 0$ iff $i = k$, namely every ω_k has no λ_k term in its expression. We can take care of this by adding to ω_k any other $\omega_{k'}$, $k' \neq k$. Hence in this case we have

$$\|\omega_I\| \leq (n-1) \frac{\max\limits_{1\leq i,j\leq n} \{\hat{\beta}^i_j\}}{\min\limits_{\substack{1\leq i,j\leq n \\ i\neq j}} \{\hat{\beta}^i_j\}} ((\omega_I)_k + (\omega_I)_{k'})$$

for all $1 \leq k, k' \leq n$, $k \neq k'$.

Finally note that $f_I = \sum_{1\leq i,j\leq n} A^i_{Ij} e_i \otimes \varepsilon^j$, so that

$$\omega_I = f_I^*(\omega) = \sum_{1\leq i,j,k\leq n} \hat{\beta}^i_j A^j_{Ik} \varepsilon^k$$

and therefore

$$(\omega_I)_k \leq \max_{1\leq i,j\leq n} \{\hat{\beta}^i_j\} \sum_{1\leq j\leq n} A^j_{Ik} = \max_{1\leq i,j\leq n} \{\hat{\beta}^i_j\} \|f_I^*(e_k)\|_{\ell^1}$$

and

$$(\min_{\hat{\beta}^i_j > 0} \hat{\beta}^i_j) \|f_I\|_{\ell^1} = (\min_{\hat{\beta}^i_j > 0} \hat{\beta}^i_j) \sum_{j,k} A^j_{Ik} \leq \|\omega_I\|_{\ell^1}$$

from which follows the claim of this lemma. □

Example 2. *Consider the Sierpinski gaskets $\boldsymbol{C}_n^\alpha$ introduced in Example 1. A proper frame for $(\boldsymbol{C}_n^\alpha)^*$ is given by $\eta^i = (\alpha-1)\varepsilon^i + \sum_{j\neq i}\varepsilon^j$, so that $\omega = \sum_{1\leq i\leq n}\eta^i = (\alpha+n-2)\sum_{1\leq i\leq n}\varepsilon^i$ and therefore*

$$\omega_I = f_I^*(\omega) = (\alpha+n-2)\sum_{1\leq i\leq n} f_I^*(\varepsilon^i) = (\alpha+n-2)\sum_{1\leq i,k\leq n} A_{Ik}^i\varepsilon^k =$$

$$= (\alpha+n-2)\sum_{1\leq k\leq n}\|f_I^*(e_k)\|_{\ell^1}\varepsilon^k.$$

If $\alpha=1$ then

$$\|\omega_I\|_{\ell^1} = \sum_{i,j}\lambda_j\hat{\beta}_i^j = (n-1)\sum_j\lambda_j$$

and

$$(\omega_I)_i = \sum_{j\neq i}\lambda_j$$

so that

$$\|\omega_I\|_{\ell^1} \leq (n-1)\left((\omega_I)_k + (\omega_I)_{k'}\right)$$

or, equivalently,

$$\|f_I\|_{\ell^1} \leq (n-1)\left(\|f_I(e_k)\|_{\ell^1} + \|f(e_{k'})\|_{\ell^1}\right)$$

for every $k\neq k'$

If $\alpha>1$ then

$$\|\omega_I\|_{\ell^1} = \sum_{i,j}\lambda_j\hat{\beta}_i^j = (n-1)\max\{\alpha-1,1\}\sum_j\lambda_j$$

and

$$(\omega_I)_i \geq \min\{\alpha-1,1\}\sum_i\lambda_j$$

so that

$$\|\omega_I\|_{\ell^1} \leq (n-1)\min\{\alpha-1,1\}(\omega_I),$$

or, equivalently,

$$\|f_I\|_{\ell^1} \leq (n-1)\min\{\alpha-1,1\}\|f_I(e_k)\|_{\ell^1},$$

for all k.

Theorem 3. *Every simple Sierpinski gasket is a fast gasket.*

Proof. The key fact here is that in every Sierpinski gasket $\boldsymbol{F}$ with a proper frame $\mathcal{E}$, for any $i \neq j$ and every k, $f_{ij}(e_k)$ is linearly dependent on both e_i and e_j. Indeed if $k \neq j$ then

$$f_{ij}(e_k) = f_i(\beta_{jk}e_j + \alpha_{jk}e_k) = \alpha_{jk}f_i(e_k) + \beta_{jk}\left(\beta_{ij}e_i + \alpha_{ij}e_j\right),$$

while if $k = j$ then

$$f_{ij}(e_j) = \alpha_{jj}(\beta_{ij}e_i + \alpha_{ij}e_j).$$

Hence the matrix representing f_{ij} with respect to $\mathcal{E}$ has at least (as matter of fact, exactly) two rows with all non-zero coefficients. This means that every column of the matrix representing f_{ijL} is a linear combination with strictly positive coefficients of $f_L(e_i)$ and $f_L(e_j)$ with possibly some other positive contribution from the other vectors.

From this we deduce immediately that

$$\|f_{IijL}\|_{\ell^1} \geq C\|f_I\|_{\ell^1}\left(\|f_L(e_i)\|_{\ell^1} + \|f_L(e_j)\|_{\ell^1}\right)$$

for some $C \geq 0$ and therefore, by Lemma 2, that

$$\|f_{IijL}\|_{\ell^1} \geq C'\|f_I\|_{\ell^1}\|f_L\|_{\ell^1}$$

for some $C' \geq 0$. Since $\|f_L\|_{\ell^1} \geq C''\|f_{ijL}\|_{\ell^1}$ for all $i \neq j$ and some $C'' > 0$, our claim follows. □

The following result is a direct consequence of Theorem 2 and 3 above and of Corollary 3 in [DeL14]:

Corollary 1. *Let $\boldsymbol{F} = \langle f_1, \dots, f_n \rangle$ be a simple Sierpinski gasket, $N_{\boldsymbol{F}}(r)$ the number of elements of $\boldsymbol{F}$ with norm smaller than r and σ_n a permutation such that f_{σ_n} are sorted in non-decreasing norm order. Then*

$$\lim_{r\to\infty} \frac{\log N_{\boldsymbol{F}}(r)}{\log r} = s_{\boldsymbol{F}} < \infty$$

and

$$\lim_{n\to\infty} \frac{\log \|f_{\sigma_n}\|}{\log n} = \frac{1}{s_{\boldsymbol{F}}}$$

In particular, this result confirms and explains the asymptotic behaviour of the norms of elements of $\boldsymbol{C}_3$ we discovered numerically (see Fig. 1). A method to evaluate analytical bounds for $s_{\boldsymbol{F}}$ is provided within Theorem 1 in [DeL14] and will be used in Section 3.2 to get bounds for $s_{\boldsymbol{C}_3}$.

Our last result on simple projective Sierpinski gaskets is that they have all zero Lebesgue measure. We recall that this is a necessary condition for the non-integrality of their Hausdorff dimension, as suggested by our numerical investigations (see next Section).

Lemma 3. *Let $\boldsymbol{F} \subset Aut(\mathbb{R}^n)$ be a simple Sierpinski gasket, $\mathcal{E}$ a proper frame of $\boldsymbol{F}$ and μ any measure of $\mathbb{R}P^{n-1}$ in the measure class of the* round measure, *namely the measure induced by the metric of sectional curvature identically equal to 1. Then there exist constants $A, B, C, D > 0$ s.t.*

$$\frac{A}{\|f_I\|^n} \leq \mu(T_{\boldsymbol{F} f_I}) \leq \frac{B}{\|f_I\|^{a_n}}, \quad \frac{C}{\|f_I\|^n} \leq \mu(Z_{\boldsymbol{F} f_I}) \leq \frac{D}{\|f_I\|^n},$$

where $a_n = n$ if $\boldsymbol{F}$ is of the second kind and $a_n = n-1$ if it is of the first kind.

Proof. It is enough to prove the claim in some chart containing $T_{\boldsymbol{F}}$. We fix coordinates $(x^1, \dots, x^n)$ so that the vectors of $\mathcal{E}$ are

$$e_1 = (1, 0, \dots, 0, 1), \dots, e_{n-1} = (0, \dots, 0, 1, 1), e_n = (0, \dots, 0, 1)$$

and use the chart $x^n = 1$. In this chart we pick any smooth measure ν of finite total volume and with constant density equal to 1 within $T_{\boldsymbol{F}}$.

Note that $f_I = A^i_{Ik} e_i \otimes \varepsilon^k = A^i_{Ik} e^j_i \partial_j \otimes \varepsilon^k$, where the last row of the matrix $A^i_{Ik} e^j_i$ contain the ℓ^1 norms of the vectors $f_I(e_i)$. A direct calculation shows that

$$\mu(T_{\boldsymbol{F} f_I}) = \frac{1}{n! \prod_{k=1}^{n} A^i_{Ik} e^n_i} = \frac{1}{n! \prod_{k=1}^{n} \|f_I(e_k)\|_{\ell^1}}.$$

Clearly $\dfrac{\prod_{k=1}^{n} \|f_I(e_k)\|_{\ell^1}}{\|f_I\|^n_{\ell^1}} \leq 1$. By Proposition 2, if $\boldsymbol{F}$ is of the second kind then

$$A \leq \frac{\prod_{k=1}^{n} \|f_I(e_k)\|_{\ell^1}}{\|f_I\|^n_{\ell^1}}$$

for some $A > 0$. If it is of the first kind assume, for the argument sake, that $\|f_I(e_1)\|_{\ell^1} \leq \ldots \|f_I(e_n)\|$. Then

$$\frac{\|f_I\|_{\ell^1}^{n-1}}{\prod_{k=1}^{n} \|f_I(e_k)\|_{\ell^1}} = \frac{\left(\sum_{k=1}^{n} \|f_I(e_k)\|_{\ell^1}\right)^{n-1}}{\prod_{k=1}^{n} \|f_I(e_k)\|_{\ell^1}} \leq$$

$$\leq C \prod_{k=1,n-1} \frac{\|f_I(e_k)\|_{\ell^1} + \|f_I(e_{k+1})\|_{\ell^1}}{\|f_I(e_{k+1})\|_{\ell^1}} \leq 2^{n-1}C$$

The geometry of $Z_{\boldsymbol{F} f_I}$ is more complex. We divide it in n $(n-1)$-simplices Z_i, where Z_i's vertices are the $(n-1)$ points $[f_i(f_I(e_j))]$, $j \neq i$, plus the point $[\sum_{1\leq i\leq n} f_I(e_i)]$. Then

$$\mu(Z_i) = \frac{1}{n!\|f_I(\sum_{1\leq i\leq n} e_i)\|_{\ell^1} \prod_{j\neq i} \|\beta_{ij}(e_I)_i + \gamma_i(e_I)_j\|_{\ell^1}} =$$

$$= \frac{1}{n!\|f_I\|_{\ell^1} \prod_{j\neq i} \|\beta_{ij}(e_I)_i + \gamma_i(e_I)_j\|_{\ell^1}}$$

since all components of all vectors are positive. Even in this case then

$$\frac{1}{\mu(Z_i)\|f_I\|_{\ell^1}^n} \leq n!$$

and

$$\frac{\|f_I\|_{\ell^1}^n}{\|f_I\|_{\ell^1} \prod_{j\neq i} \|\beta_{ij}(e_I)_i + \gamma_i(e_I)_j\|_{\ell^1}} \leq \frac{\|f_I\|_{\ell^1}^{n-1}}{\max\{\beta, \gamma\} \prod_{j\neq i} \|f_I(e_i) + f(e_j)\|_{\ell^1}} \leq A$$

for some A. $\square$

Theorem 4. *The attractor of a simple Sierpinski gasket $\boldsymbol{F} \subset SL_n^{\pm}(\mathbb{R})$ is a null set with respect to the measure class of the round measure on $\mathbb{R}P^{n-1}$.*

Proof. From the previous lemma we see that

$$1 \leq \frac{\mu(T_{\boldsymbol{F} f_I})}{\mu(Z_{\boldsymbol{F} f_I})} \leq \frac{C}{\|f_I\|}$$

if $\boldsymbol{F}$ is of the first kind and

$$1 \leq \frac{\mu(T_{\boldsymbol{F} f_I})}{\mu(Z_{\boldsymbol{F} f_I})} \leq C$$

if it is of the second. Let us first assume that $\boldsymbol{F}$ is of the second kind and let

$$S_k = \sum_{|I|=k+1} \mu(T_{\boldsymbol{F} f_I}) \text{ and } P_k = \sum_{|I|=k} \mu(Z_{\boldsymbol{F} f_I}).$$

Then

$$S_{k+1} = S_k - P_k \leq S_k(1 - C)$$

and therefore

$$S_k - S_{k+1} \geq C S_k$$

so that, after making a telescopic sum, we get

$$S_1 - \lim_{k\to\infty} S_k \geq C \lim_{k\to\infty} k S_k$$

which immediately implies that $\lim_{k\to\infty} S_k = 0$.

If $\boldsymbol{F}$ is of the first kind then $\mu(T_{\boldsymbol{F} f_I})/\mu(Z_{\boldsymbol{F} f_I}) \leq C/\min_{|J|=|I|} \|f_J\|$. It is easy to check that $\min_{|J|=|I|} \|f_J\|$ is proportional to $|I|$ and therefore

$$S_{k+1} = S_k - P_k \leq S_k(1 - C/k)$$

and

$$S_k - S_{k+1} \geq C S_k / k$$

so that

$$S_1 - \lim_{k\to\infty} S_k \geq C \lim_{k\to\infty} S_k \sum_{1\leq j\leq k} 1/j.$$

Since the series $1/j$ diverges we get again that $\lim_{k\to\infty} S_k = 0$. □

3 Numerical Results

All numerical results presented in this section were obtained through Perl and C programs written by the author. Most of the results were obtained through the following three elementary routines: 1) generate all triangles (or circles) belonging to $T_{k,\boldsymbol{F}}$ given a simple Sierpinski (or Apollonian) gasket $\boldsymbol{F}$ and an integer k; 2) generate *all* matrices of $\boldsymbol{F}$ with norm not larger than k; 3) evaluate the number of squares of a given side length needed to cover the complement of $T_{k,\boldsymbol{F}}$.

3.1 Affine Sierpinski Gaskets

In [FL98] Falconer and Lammering studied in detail the family of affine Sierpinski gaskets $S_{a,b}$, $a, b \in (0,1)$, defined by the affine transformations

$$S_1 \begin{pmatrix} x \\ y \end{pmatrix} = (1-a) \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ a \end{pmatrix}$$
$$S_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
$$S_3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1-b & 1-a-b \\ 0 & a \end{pmatrix} + \begin{pmatrix} b \\ 0 \end{pmatrix}.$$

Notice that the case $a = \frac{1}{2}, b = \frac{1}{2}$ corresponds to the standard Sierpinski gasket. In particular they proved that the box dimension of the corresponding attractor $R_{a,b}$ is given by the unique root of the equation

$$(1-a)^s + ab^{s-1} + a(1-b)^{s-1} = 1 \tag{3}$$

in the triangle $T_1 = \{(a,b) \in (0,1)^2, a \geq \max\{b, 1-b\}\}$ and of

$$(1-a)^s + a^{s-1} = 1 \tag{4}$$

in the opposite triangle $T_2 = \{(a,b) \in (0,1)^2, a \leq \min\{b, 1-b\}\}$. This setting provides a convenient source of examples for comparing the box dimension of the attractor of a real projective Sierpinski gasket with the corresponding gasket exponent. Indeed the injection sending the affine transformation

$S(x) = Tx + v$, with $T \in M_2(\mathbb{R})$ and $x, v \in \mathbb{R}^2$, into the 3×3 matrices $\begin{pmatrix} T & v \\ 0 & 1\end{pmatrix}$, applied to the S_i, gives the three matrices

$$M_1 = \begin{pmatrix} 1-a & 0 & 0 \\ 0 & 1-a & a \\ 0 & 0 & 1\end{pmatrix}, M_2 = \begin{pmatrix} b & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1\end{pmatrix}, M_3 = \begin{pmatrix} 1-b & 1-a-b & b \\ 0 & a & 0 \\ 0 & 0 & 1\end{pmatrix},$$

so that the action induced by the M_i on $\mathbb{R}P^2$ in the affine chart $z = 1$ coincides with the action of the S_i.

These matrices are all upper triangular and $\|M_I\| = 1$ for all $I \in \mathcal{I}^3$. To see this, first of all we let

$$M = \begin{pmatrix} \alpha & \lambda & \mu \\ 0 & \beta & \nu \\ 0 & 0 & 1\end{pmatrix}$$

and notice that $0 \le \lambda + \beta \le 1$ and $0 \le \mu + \nu \le 1$. Indeed we can limit the discussion to left multiplication by M_1 and M_3 and it is immediate to verify by induction that, assuming the inequalities above for M_I, the products

$$M_1 M = \begin{pmatrix} (1-a)\alpha & (1-a)\lambda & (1-a)\mu \\ 0 & (1-a)\beta & (1-a)\nu + \alpha \\ 0 & 0 & 1\end{pmatrix}$$

and

$$M_3 M = \begin{pmatrix} (1-b)\alpha & (1-b)\lambda + (1-a-b)\beta & (1-b)\mu + (1-a-b)\nu + b \\ 0 & a\beta & a\nu \\ 0 & 0 & 1\end{pmatrix}$$

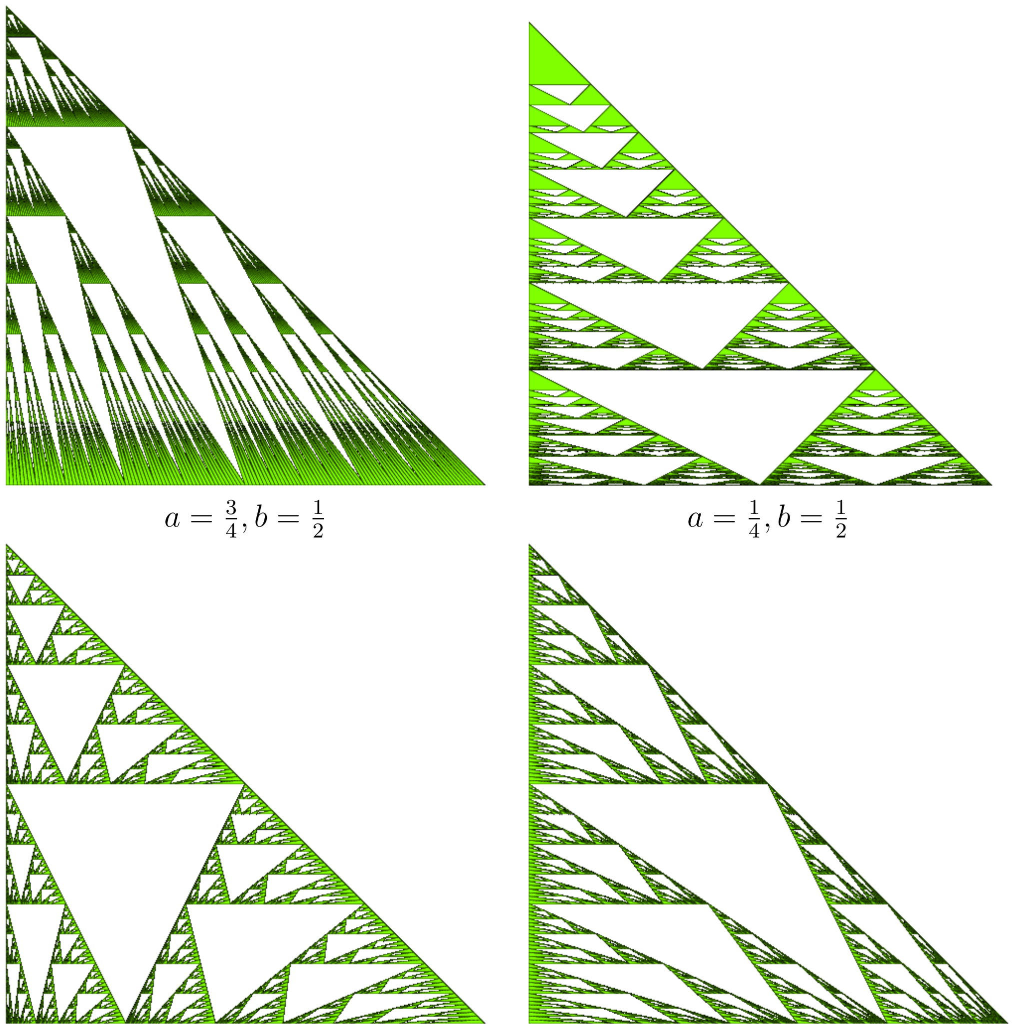
satisfy the same inequalities. Now, consider the semigroup $\boldsymbol{A}_{a,b}$ generated by the $A_i = M_i / \det M_i^{1/3} \in SL_3(\mathbb{R})$. Clearly $\|A_I\| = \det M_{i_1}^{-1/3} \cdots \det M_{i_k}^{-1/3}$ for every $I = i_1 \cdots i_k$ and therefore

$$\zeta_{\boldsymbol{A}_{a,b},k} = \sum_{I \in \mathcal{I}^3_k} \|A_I\|^{-s} = \left(\det M_1^{s/3} + \det M_2^{s/3} + \det M_3^{s/3}\right)^k.$$

Then, by Theorem 2 in [DeL14], the exponent $s_{\boldsymbol{A}_{a,b}}$ is the unique solution of the equation

$$(1-a)^{2s/3} + (ab)^{s/3} + (a(1-b))^{s/3} = 1. \tag{5}$$

Next proposition shows that Conjecture 1 holds for the affine Sierpinski gaskets $\boldsymbol{A}_{a,b}$ when (a, b) lies within $T_1 \cup T_2$:



$a = \frac{3}{4}, b = \frac{1}{2}$ $a = \frac{1}{4}, b = \frac{1}{2}$

$a = \frac{1}{2}, b = \frac{1}{4}$ $a = \frac{1}{2}, b = \frac{3}{4}$

Figure 4: Affine Sierpinski gaskets $\boldsymbol{A}_{a,b}$ for four possible pairs a, b. For each one we plot (in green) the set $T_{7,\boldsymbol{A}_{a,b}}$. For the upper two the box dimensions can be evaluated analytically and are equal, with an error of 1 on the last digit, to, respectively, $\dim_B R_{\frac{3}{4},\frac{1}{2}} = 1.72368$ and $\dim_B R_{\frac{1}{4},\frac{1}{2}} = 1.68886$. The numerical evaluation of the box dimensions with an elementary box-counting algorithm gives: $\dim_B R_{\frac{3}{4},\frac{1}{2}} \simeq 1.71$, $\dim_B R_{\frac{1}{4},\frac{1}{2}} \simeq 1.66$, $\dim_B R_{\frac{1}{2},\frac{1}{4}} \simeq 1.60$, $\dim_B R_{\frac{1}{2},\frac{3}{4}} \simeq 1.60$. See Table 1 for a comparison of the box dimension of these and other affine gaskets with the exponent of the corresponding real projective Sierpinski gaskets.

Proposition 1. *The inequality* $3\dim_B R_{a,b} \geq 2s_{\boldsymbol{A}_{a,b}}$ *holds within the two triangles* $T_{1,2}$, *with the equal sign holding only in their common vertex.*

Proof. After writing (5) in terms of $t = 2s/3$ and renaming t to s we are left with the equation $(1-a)^s + (ab)^{s/2} + (a(1-b))^{s/2} = 1$. Comparing this expression with the left-hand sides of (3) and (4) we see that it is enough to prove that

$$(ab)^{s/2} + (a(1-b))^{s/2} \leq \min\{a^{s-1}, ab^{s-1} + a(1-b)^{s-1}\}.$$

Since, for obvious geometrical reasons, $\dim_B R_{a,b} \leq 2$ we can assume in the following $s \geq 2$. Let us denote, respectively, by $f_{a,b}(s), g_a(s), h_{a,b}(s)$ the three functions above and notice that, since by hypothesis $a, b, 1-b \in (0,1)$, they are all strictly monotonically decreasing functions of s converging to 0 as $s \to \infty$. Moreover $f_{a,b}(2) = g_a(2) = h_{a,b}(2) = a$ and since these functions can have only one intersection it is enough to verify their behaviour for $s \to 0$. A direct calculation shows that, for every pair $a, b \in (0,1)^2$, we have that $\lim_{s\to 0} f_{a,b}(s) = 2$ while $\lim_{s\to 0} g_a(s) = \lim_{s\to 0} h_a(s) = \infty$. $\square$

We performed numerical investigations for seven different pairs (a, b), four of which belonging to $T_1 \cup T_2$. The picture of the seventh order approximation $T_{7,\boldsymbol{A}_{a,b}}$ of four of them is shown in Fig. 4, the numerical data for all of them is reported in Table 1. The critical exponent $s_{\mathcal{A}_{a,b}}$ have been evaluated analytically from Eq. (5) and compared, as a check for our numerical routines, with their numerical estimate obtained by evaluating the function $N_{\mathcal{A}_{a,b}}(k)$ for $k = 2^p$, $p = 1, \dots, 12$, and then best-fitting the points $(p, \log_2 N_{\mathcal{A}_{a,b}}(2^p))$ with a straight line – in the worst case, the numerical evaluation is less than 3% far from the analytical value. The box dimension of the attractors $R_{\boldsymbol{A}_{a,b}}$ has been evaluated analytically from Eqs. (3,4) for the pairs $(a, b) \in T_1 \cup T_2$ and numerically for all pairs via an elementary box-counting routine – in the worst case, the numerical evaluation is less than 6% far from the analytical value. Since we have no reason to believe that our numerical estimates for $\dim_B R_{\boldsymbol{A}_{a,b}}$ lose precision outside of $T_1 \cup T_2$, our numerical results lead us to believe that our Conjecture 1 holds for all $(a, b) \in (0,1)^2$, with the strict inequality sign holding everywhere except at $a = b = 1/2$, when the two quantities coincide. Moreover it appears that, roughly, $\frac{2}{3}s_{\boldsymbol{A}_{a,b}} \geq \frac{9}{10}\dim_B R_{\boldsymbol{A}}$.

a	b	$s_{\boldsymbol{A}}$ (num.)	$s_{\boldsymbol{A}}$ (anal.)	$2s_{\boldsymbol{A}}/3$	$\dim_B R_{\boldsymbol{A}}$ (num.)	$\dim_B R_{\boldsymbol{A}}$ (anal.)
1/4	1/2	2.44	2.42632	1.61755	1.66	1.68886
3/4	1/2	2.48	2.45425	1.63617	1.71	1.72368
1/2	3/4	2.35	2.34443	1.56295	1.60	–
1/5	3/10	2.44	2.43735	1.62490	1.76	1.71262
4/5	3/10	2.47	2.46960	1.64640	1.77	–
3/10	1/5	2.43	2.37354	1.58236	1.75	–
7/10	1/5	2.35	2.35249	1.56833	1.72	1.63373

Table 1: Values of the exponents of affine Sierpinski gaskets $\boldsymbol{A}_{a,b}$ for several pairs a, b and of the box dimension of the corresponding attractors $R_{a,b}$. Numerical evaluations for $s_{\boldsymbol{A}}$ were done by calculating $N_{\boldsymbol{A}}(k)$ for the values $k = 2^p$, $p = 1, \dots, 12$, and are presented to motivate our confidence in a relative error not bigger than 1% in the other evaluations provided throughout the paper when an analytical evaluation is not available. Numerical evaluations for the box dimension of $R_{a,b}$ were done via an elementary box-counting algorithm and a comparison with the available analytical evaluations suggest that their relative error is no more than 6%.

3.2 The cubic semigroups $\boldsymbol{C}_n^\alpha$

Recall that, by definition, $\boldsymbol{C}_n^\alpha$ is generated by the n linear maps $f_i^\alpha(e_i) = \alpha e_i$, $f_i^\alpha(e_j) = e_j + e_i$, $j \neq i$.

$\boldsymbol{n = 3}$. In $\mathbb{R}^3$ we use coordinates (x, y, z) with respect to the frame $e'_1 = e_1 + e_3$, $e'_2 = e_2 + e_3$, $e'_3 = e_3$, so that the f_i are represented by the matrices

$$A_1 = \begin{pmatrix} \alpha - 1 & 0 & 1 \\ 0 & 1 & 0 \\ \alpha - 2 & 0 & 2 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha - 1 & 1 \\ 0 & \alpha - 2 & 2 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 - \alpha & 2 - \alpha & \alpha \end{pmatrix}.$$

In the affine chart $[x : y : z] \to (u, v) = (x/z, y/z)$ of $\mathbb{R}P^2$ we have therefore that

$$\begin{cases} \psi_1(x, y) = \left(\frac{(\alpha-1)u+1}{(\alpha-2)u+2}, \frac{v}{(\alpha-2)u+2}\right) \\ \psi_2(x, y) = \left(\frac{u}{(\alpha-2)v+2}, \frac{(\alpha-1)v+1}{(\alpha-2)v+2}\right) \\ \psi_3(x, y) = \left(\frac{u}{(2-\alpha)(u+v)+\alpha}, \frac{v}{(2-\alpha)(u+v)+\alpha}\right) \end{cases}$$

α	$s_{\boldsymbol{C}_3^\alpha}$	$2s_{\boldsymbol{C}_3^\alpha}/3$	$\dim_B R_{\boldsymbol{C}_3^\alpha}$
1	2.447	1.631	1.72
1.3	2.395	1.596	1.72
1.7	2.377	1.585	1.71
2	2.359	1.573	1.59
3	2.378	1.586	1.71
4	2.389	1.593	1.73
7	2.394	1.596	1.76

Table 2: Numerical evaluation of the exponent of the real projective gaskets $\boldsymbol{C}_3^\alpha$ and of the box dimension of the corresponding attractors for several values of α. No analytical formula is known for these quantity. These data confirms the relation $2s_{\boldsymbol{A}}/3 \leq \dim_B R_{\boldsymbol{A}}$ already observed in Table 1 and the fact that roughly $2s_{\boldsymbol{A}}/3 \geq 9\dim_B R_{\boldsymbol{A}}/10$.

and the vertices of the invariant triangle $T_{\boldsymbol{C}_3^\alpha}$ are $[e_1] = (1,0)$, $[e_2] = (0,1)$ and $[e_3] = (0,0)$. A direct calculation of the eigenvalues of the Jacobian matrices $D\psi_i$ shows that, within $T_{\boldsymbol{C}_3^\alpha}$,

$$\min\{\frac{1}{\alpha},\frac{\alpha}{4}\}d(x,y) \leq d(\psi_i(x),\psi_i(y)) \leq \max\{\frac{1}{\alpha},\frac{\alpha}{4}\}d(x,y)$$

for all $i = 1,2,3$, namely the semigroup $\langle\psi_1,\psi_2,\psi_3\rangle$, *as a IFS*, is hyperbolic for $\alpha \in (1,4)$ and parabolic for $\alpha = 1,4$. The ψ_i are not contractions *with respect to the Euclidean distance in this chart* for higher values of α (see Fig. 5 for the plot of $T_{7,\boldsymbol{C}_3^\alpha}$ for several values of α).

Analytical bounds for the Hausdorff dimension of the attractors $R_{\boldsymbol{C}_3^\alpha}$ can be obtained via Propositions 9.6 and 9.7 in [Fal90], namely

$$\min\{\frac{1}{\log_3\frac{4}{\alpha}},\frac{1}{\log_3\alpha}\} \leq \dim_H R_{\boldsymbol{C}_3^\alpha} \leq \max\{\frac{1}{\log_3\frac{4}{\alpha}},\frac{1}{\log_3\alpha}\}.$$

For $\alpha = 2$ we get, as expected, $\dim_B R_{S_3} = \dim_H R_{S_3} = \log_2 3$. Unfortunately these bounds get quickly too loose to be significant for us as soon as $|\alpha-2| > 10^{-1}$. Our numerical evaluations of $\dim_B \boldsymbol{C}_3^\alpha$ and $s_{\boldsymbol{C}_3^\alpha}$ for several values of α (see Table 2) indicate, though, that Conjecture 1 holds even for these gaskets. Moreover, like for the affine gaskets, the box dimension seems to not to fall further than a 10% away from the quantity $2s_{\boldsymbol{C}_3^\alpha}/3$.

Analytical bounds for the exponents $s_{\boldsymbol{C}_3^\alpha}$ can be obtained from Theorem 1 in [DeL14]. Here we present calculations for $\boldsymbol{C}_3$, the Levitt-Yoccoz gasket.

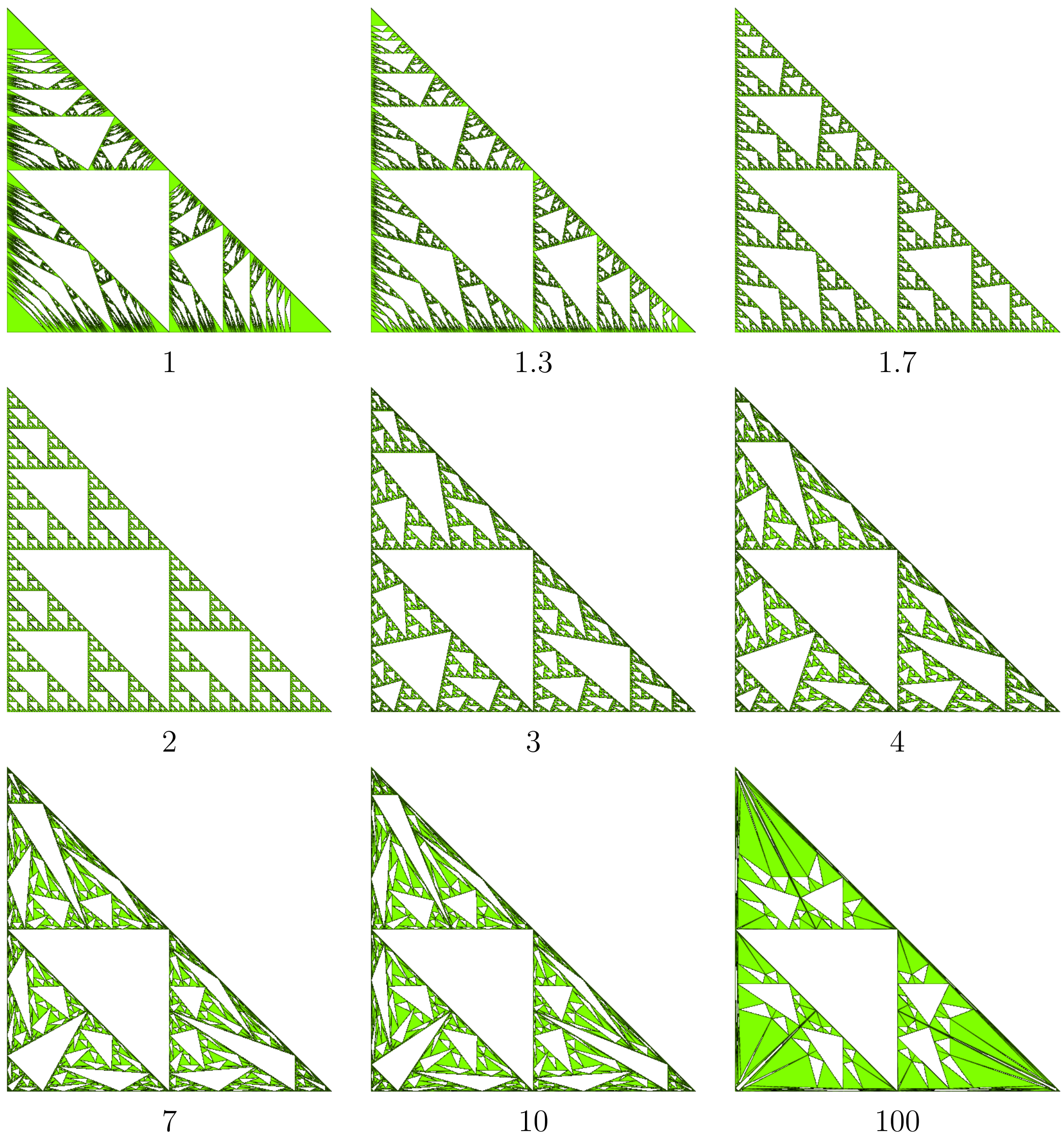


Figure 5: Real projective Sierpinski gaskets $\boldsymbol{C}_3^\alpha$ for several values of α. For each one we plot (in green) the set $T_{7,\boldsymbol{C}_3^\alpha}$. Heuristic numerical estimates of their exponents and of the box dimension for the corresponding attractors for $\alpha \leq 7$ are given in Table 2

Due to the symmetry between the generators it turns out that

$$\mu_{\boldsymbol{C}_3}(s) = 6\mu_{\boldsymbol{C}_3 A_{12}} = 3 \cdot 2^{1-s}\zeta(s),$$

from which, as the unique solution of $\mu_{\boldsymbol{C}_3}(s) = 3^s$, we get the lower bound $1.52 \leq s_{\boldsymbol{C}_3}$. To get the first upper bound we must consider the function

$$\mu_{\boldsymbol{C}_3,2}(s) = 3 \cdot 2^{1-s}\left(3\zeta(s) + 2^{2-s}\zeta(s, \frac{7}{4}) - 2^{1-s} - 3\right),$$

from which we get $1.7 \leq s_{\boldsymbol{C}_3} \leq 7.1$ as the unique solutions of $\mu_{\boldsymbol{C}_3,2}(s) = 3^{\pm s}$. In order to get more meaningful bounds we should consider some $\mu_{\boldsymbol{C}_3,k}$ with a large k but leave this to a future paper. By best-fitting with a straight line the curve $\log N_{\boldsymbol{C}_3}(k)$ as function of $\log k$ for $k = 2^p$, $1 \leq p \leq 13$, we get a reliable estimate of $s_{\boldsymbol{C}_3} \simeq 2.4438$. A rough numerical evaluation of the box dimension of $R_{\boldsymbol{C}_3}$ by counting the number of squares needed to cover the fractal gives $\dim_B \simeq 1.72$, compatible with the relation $3 \dim_B R_{\boldsymbol{C}_3} \geq 2 s_{\boldsymbol{C}_3}$ suggested in Conjecture 1.

$\boldsymbol{n \geq 4}$**.** In $\mathbb{R}^n$ we use coordinates $(x^1, \ldots, x^n)$ with respect to the frame $e'_1 = e_1 + e_n, \ldots, e'_{n-1} = e_{n-1} + e_n, e'_n = e_n$. For $n = 4$ the matrices A_1 and A_4 are given by

$$A_1 = \begin{pmatrix} \alpha - 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \alpha - 2 & 0 & 0 & 2 \end{pmatrix}, A_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 - \alpha & 2 - \alpha & 2 - \alpha & \alpha \end{pmatrix}$$

and A_2 and A_3 can be obtained from A_1 via permutations of the coordinates. Similarly, for $n > 4$, A_1 and A_n are the natural generalization of the matrices above and the A_i, $1 < i < n$, are obtained through permutations of the coordinates. Correspondingly, in $\mathbb{RP}^{n-1}$ we use coordinates $u^i = x^i/x^n$, $i = 1, \ldots, n-1$, and obtain

$$\psi_1(u^i) = \left(\frac{(\alpha-1)u^1 + 2}{(\alpha-1)u^1 + 2}, \frac{u^2}{(\alpha-1)u^1 + 2}, \ldots, \frac{u^{n-1}}{(\alpha-1)u^1 + 2}\right),$$

similarly for $i < n - 1$ and

$$\psi_{n-1}(u^i) = \left(\frac{u^1}{(2-\alpha)(u^1 + \cdots + u^{n-1}) + \alpha}, \ldots, \frac{u^{n-1}}{(2-\alpha)(u^1 + \cdots + u^{n-1}) + \alpha}\right).$$

A direct evaluation of the eigenvalues of the Jacobian matrices of the ψ_i gives the same result we got for $n = 3$, namely for every $n \geq 3$ we have that the gasket $\boldsymbol{C}_n^\alpha$ is a hyperbolic IFS for $\alpha \in (1, 4)$ and a parabolic IFS for $\alpha = 1, 4$. The bounds on the Hausdorff dimension of the attractors give

$$\min\{\frac{1}{\log_n \frac{4}{\alpha}}, \frac{1}{\log_n \alpha}\} \leq \dim_H R_{\boldsymbol{C}_n^\alpha} \leq \max\{\frac{1}{\log_n \frac{4}{\alpha}}, \frac{1}{\log_n \alpha}\}.$$

For $n = 4, \alpha = 2$ we get the well-known result that the dimension of the standard Sierpinski tetrahedron is equal to 2. Numerical evaluations of the box-counting dimension of $\boldsymbol{C}_4 \stackrel{\text{def}}{=} \boldsymbol{C}_4^1$ suggest that the same could hold for the 4-dimensional version of the cubic gasket (see Fig. 6 for a picture of the two sets). This is compatible with the numerical evaluation of the critical exponent $s_{\boldsymbol{C}_4} \simeq 2.758$ from the data for $N_{\boldsymbol{C}_4}(k)$ shown in Table 3. From it indeed we get that $\dim_H \boldsymbol{C}_4 \geq 3s_{\boldsymbol{C}_4}/4 \simeq 2.07$, that allows the value $\dim_H \boldsymbol{C}_4 = 2$ if we assume an error of about 4% on $s_{\boldsymbol{C}_4}$, a precision in agreement with the one measured in the numerical evaluations made above for the affine gaskets.

3.3 The Apollonian gasket

We conclude the paper with a brief discussion on the Apollonian semigroup, namely the semigroup $\boldsymbol{H} \subset SL_4(\mathbb{N})$ generated by the matrices H_1, H_2, H_3 defined in the introduction. This case was thoroughly studied, somehow implicitly, by Boyd, in particular in [Boy72, Boy73b, Boy82], in the context of the sequence of curvatures in an Apollonian gasket. Boyd's investigation and arguments were the archetype for most results and arguments in Section 2 of [DeL14].

We start by proving that, in our context, the reason for the asymptotic behaviour shown in Fig. 2 is the same that holds for the Levitt-Yoccoz gasket, namely that the free semigroup $\boldsymbol{H}$ is fast.

Lemma 4. *Assume that matrices $A_1, \dots, A_m \in M_n(\mathbb{N})$ have the following properties:*

1. *they have some number $k \neq 1$ of rows containing a single entry equal to 1 and all other equal to 0 and these entries equal to 1 belong all to different columns and in those columns all entries are either 0 or 1;*

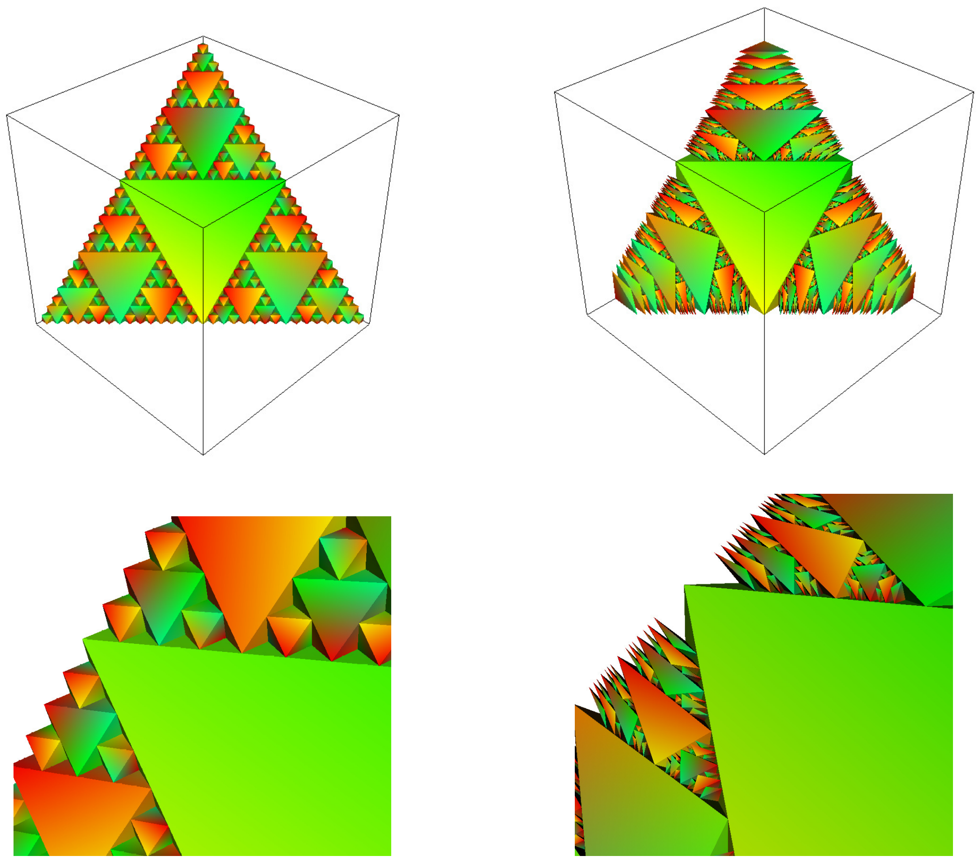


Figure 6: Images of the Sierpinski ($\boldsymbol{S}_4 = \boldsymbol{C}_4^2$) and the cubic ($\boldsymbol{C}_4 = \boldsymbol{C}_4^1$) tetrahedra. In figure we show a full picture (above) and a detail (below) for the sets $T_{5,\boldsymbol{S}_4}$ (left) and $T_{5,\boldsymbol{C}_4}$ (right).

2. *other rows are such that each of their entries is smaller than the sum of the remaining $n-1$ entries.*

Then this property is shared by all products of the A_i.

Proof. We prove the lemma by induction. It is enough to consider the products of two generic matrices $A = (A^i_j)$, $B = (B^i_j)$, satisfying the hypotheses.

Assume first that $k = 0$ for B, namely $\sum_{k\neq\ell} B^i_k \geq B^i_\ell$ for all i, ℓ. Then

$$\sum_{k\neq\ell}(AB)^i_k = \sum_{\substack{1\leq j\leq n\\ k\neq\ell}} A^i_j B^j_k = \sum_{1\leq j\leq n} A^i_j \sum_{k\neq\ell} B^j_k \geq \sum_{1\leq j\leq n} A^i_j B^j_\ell = (AB)^i_\ell.$$

Assume now that $k > 1$ for B and denote by $I = (i_1, \dots, i_k)$ the rows with a 1 and all other entries equal to 0. Every line (if any) of A with a 1 and all other entries equal to 0 leaves unaltered the corresponding row in B and therefore the new line satisfies the conditions in the theorem. Otherwise we notice that

$$\sum_{k\neq\ell}(AB)^i_k = \sum_{\substack{1\leq j\leq n\\ k\neq\ell}} A^i_j B^j_k \geq \sum_{\substack{1\leq j\leq n\\ j\notin I}} A^i_j B^j_\ell + \sum_{\substack{1\leq j\leq n\\ j\in I}} A^i_j \sum_{k\neq\ell} B^j_k.$$

If $j \in I$ then $\sum_{k\neq\ell} B^j_k$ is either 0 or 1. Since by hypothesis there are at least two such rows and $\sum_{k\neq\ell} B^j_k + \sum_{k\neq\ell} B^{j'}_k \geq 1$ for every $j, j' \in I$, $j \neq j'$, and the corresponding entries A^i_j and $A^i_{j'}$ are both equal to 1, then

$$\sum_{\substack{1\leq j\leq n\\ j\in I}} A^i_j \sum_{k\neq\ell} B^j_k \geq \sum_{\substack{1\leq j\leq n\\ j\in I}} A^i_j B^j_\ell = (AB)^i_\ell,$$

therefore $\sum_{k\neq\ell}(AB)^i_k \geq (AB)^i_\ell$. □

Theorem 5. *The Apollonian semigroup $\boldsymbol{H}$ is a fast gasket with coefficient $c \geq 1/4$.*

Proof. Note first of all that Hirst matrices satisfy previous Lemma's conditions. Moreover the entries in the third line are not smaller than all other entries in the same column and it is easy to see by induction that this property is preserved by products.

Let $\|A\|_\infty = \max_{1\leq i\leq n} \sum_{1\leq j\leq n} |A^i_j|$. A look at the 6 matrices H_{ij}, $i \neq j$, shows that their third column has always at least three non-zero entries, so that $\|A_{IJ}\|_\infty \geq \|A_J\|_\infty \sum_{j\neq j_0} |A^i_3|$ where j_0 is the index of the element of the third column (if any) equal to zero (otherwise just set $j_0 = 1$). By the previous Lemma and the fact that the norm of every $A \in \boldsymbol{H}$ in concentrated in the third row, the sum of any three entries of the third row of A is always larger than $\|A\|$, so that $\|A_{IJ}\|_\infty \geq \|A_J\|_\infty \|A_I\|$. Since $4\|A\| \geq \|A\|_\infty \geq \|A\|$, the claim follows. $\square$

Together with Theorem 2, this provides an independent proof, based on general principles rather than on the particular geometry of the Apollonian gasket, of point 1 of Boyd's Theorem 1.

Let us turn now to the evaluation of the critical exponent $s_{\boldsymbol{H}}$. Analytical bounds for it were studied in detail by Boyd in [Boy70, Boy72, Boy73a] and we do not attempt to improve them here. Increasingly accurate *numerical* evaluations of $s_{\boldsymbol{H}}$ with several different techniques have been given over the last half-century by Melzak [Mel69], Boyd [Boy82], Manna and Herrmann [MH91], Thomas and Dhar [TD94] and McMullen [McM98], giving respectively the following values, with a *heuristic* error of 1 unit on the last digit: 1.306951, 1.3056, 1.30568, 1.30568673, 1.305688. We remark that, among all these evaluations, the one with the largest number of digits, given by Thomas and Dhar, is the only one based on a *heuristic* method, while the others are based on *exact* methods.

Partly to test our own software evaluating the function $N_{\boldsymbol{H}}(k)$ for a generic gasket $\boldsymbol{H}$ and partly because the computational power of computers increased quite a lot over the last fifteen years, which is how old is the last evaluation of the exponent, we repeated the elementary evaluation made by Boyd in 1982 by evaluating $N_{\boldsymbol{H}}(k)$ for $k = 2^p$, $p = 1, \dots, 39$ with respect to the norm $\|A_I\| = \sum_{1\leq i,j\leq 4} (A_I)_{ij} v^i w^i$, where $v = (-1, 2, 2, 0)$ and $w = (1, 1, 1, 2)$ (this way $\|A_I\|$ is equal to the the curvature of the circle of multi-index I in the Apollonian gasket generated by the circles of radius $-1, 2, 2$), and then best-fitting with a straight line the data obtained (see Table 3). We found a value of $s_{\boldsymbol{H}} \simeq 1.3056867$ which fully confirms the heuristic evaluation of Thomas and Dhar and suggests an error of 2 on the last digit of the estimate by McMullen.

Acknowledgments

I am grateful to M.F. Barnsley, D.W. Boyd, I.A. Dynnikov, K. Falconer, B. Hunt and S.P. Novikov for several precious insights and suggestions on the

$\boldsymbol{C}_2$ (19) 2.000001	1, 7, 35, 143, 615, 2455, 9915, 39639, 159191, 636903, 2549123, 10195943, 40795027, 163182291, 652741999, 2610964315, 10444080375, 41776267143, 167105597331
$\boldsymbol{C}_3$ (15) 2.4438	4, 22, 148, 760, 4594, 24646, 136372, 740650, 4046188, 22022770, 119929126, 652445212, 3550689778, 19318095256, 105103180192
$\boldsymbol{C}_4$ (14) 2.758	5, 37, 293, 2197, 15125, 103669, 714245, 4849045, 32901077, 222724789,1507986917, 10202765749, 69029614661, 466938773125
$\boldsymbol{A}_3$ (29) 2.60	3, 3, 5, 12, 29, 64, 123, 255, 594, 1372, 3222, 7388, 17636, 41302, 95748, 226607, 534766, 1271425, 3032945, 7221236, 17258732, 41170226, 98428382, 235630679, 563928974, 1351059074, 3238344644, 7765525872, 18627071753
$\boldsymbol{H}$ (39) 1.3056867	0, 1, 3, 8, 18, 48, 113, 278, 681, 1722, 4238, 10488, 25927, 64086, 158266, 391062, 967315, 2390800, 5909752, 14608522, 36115118, 89275994, 220684802, 545546400, 1348603780, 3333755028, 8241076212, 20372155276, 50360227721, 124491161884, 307744098990, 760747405278, 1880578271904, 4648814463680, 11491932849933, 28408221038996, 70225503797745, 173598409768852, 429137646728801

Table 3: Values of $N_{\boldsymbol{A}}(a^k)$, for $k = 1, 2, \ldots, K$, for the *cubic semigroups* $\boldsymbol{C}_i$, $i = 2, 3, 4$, the Apollonian semigroup $\boldsymbol{A}_3$ and the Hirst semigroup $\boldsymbol{H}$, where $a = 1.4$ in case of $\boldsymbol{A}_3$ and $a = 2$ for all others. For each semigroup, we report in the left column the largest value K of the exponent k for which we evaluated $N_{\boldsymbol{A}}(a^k)$ (in parenthesis) and the slope of the straight line best fitting the points $(k, \log_a N_{\boldsymbol{A}}(k))$, with a heuristic error of 1 on the last digit.

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