Weak solutions of the cohomological equation on $\mathbb{R}^2$ for regular vector fields

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Abstract

In a recent article [4], we studied the global solvability of the so-called cohomological equation $L_\xi f = g$ in $C^\infty(\mathbb{R}^2)$, where ξ is a regular vector field on the plane and L_ξ the corresponding Lie derivative operator. In a joint article with T. Gramchev and A. Kirilov [5], we studied the existence of global L^1_{loc} weak solutions of the cohomological equation for planar vector fields depending only on one coordinate. Here we generalize the results of both articles by providing explicit conditions for the existence of global weak solutions to the cohomological equation when ξ is intrinsically Hamiltonian or of finite type.

1 Introduction

The topological structure of *regular* (i.e. without zeros) vector fields ξ on $\mathbb{R}^2$ has been thoroughly investigated during the last century and is well understood (see [4] for detailed references). The global analytic properties of the corresponding partial differential operators L_ξ (Lie derivative along the flow of ξ) are, on the contrary, much less known. Some of their most basic properties were studied in [4].

The main purpose of this article is to refine some of the results in [4], concerning the action of the operators L_ξ on spaces of differentiable functions, and to use them to generalize to a much wider set of regular vector fields the results obtained in [5], concerning weak solutions of the cohomological equation

$$L_\xi f = g \in C^k(\mathbb{R}^2),\ k = 1, 2, \dots, \infty \tag{1}$$

for regular planar vector fields depending only on one variable.

2 Definitions and main results

The following definitions and notations will be used in the present article.
Vector Fields and Foliations. We will usually denote vector fields by ξ and their corresponding (local) flows by ϕ^t_ξ. We say that a C^1 function F is *regular* if its differential dF is never zero; analogously, we say that a vector field is regular when it has no zeros. We denote by $\mathfrak{X}_r(\mathbb{R}^2)$ the set of all smooth regular vector fields on the plane. Given any $\xi \in \mathfrak{X}_r(\mathbb{R}^2)$, $\mathcal{F}_\xi$ will denote the smooth foliation of its integral trajectories and, by abuse of notation, the space of leaves endowed with its canonical quotient smooth structure[1]. We denote by $\pi_\xi : \mathbb{R}^2 \to \mathcal{F}_\xi$ the canonical projection that sends a point to the leaf passing through it. A *saturated* neighborhood of a leaf ℓ of $\mathcal{F}_\xi$ is a set $\pi_\xi^{-1}(U)$, where U is a neighborhood of ℓ in $\mathcal{F}_\xi$. We say that two integral trajectories s_1, s_2 of ξ are *inseparable* when they are inseparable as points in the quotient topology of $\mathcal{F}_\xi$, i.e. when the intersection of every two saturated neighbourhoods of s_1 and s_2 is non-empty. An integral trajectory s which is inseparable from some other integral trajectory is said a *separatrix*. We denote by $\mathcal{S}_\xi$ the set of all separatrices of ξ.

Definition 1. *A vector field $\xi \in \mathfrak{X}_r(\mathbb{R}^2)$ (and, by extension, the foliation $\mathcal{F}_\xi$) is of* finite type *if $\mathcal{S}_\xi$ is closed in $\mathcal{F}_\xi$ and every separatrix is inseparable from just finitely many other integral trajectories.*

Example 1. *The lines $y = \pm 1$ are the only inseparable integral trajectories of the polynomial regular planar vector field $\xi(x, y) = (2y, 1 - y^2)$ (see Fig. 3). In particular, ξ is of finite type.*

The set of vector fields of finite type is of great relevance since important categories of vector fields belong to it. For example, every planar regular polynomial vector field is of finite type: finite bounds for the number of separatrices of a such vector fields were found first by Markus [11] and later improved independently by M.P. Muller [14] and S. Schecter and M.F. Singer [16]. It is easy to verify that are of finite type also all planar regular vector fields whose foliation is invariant with respect to translations in a given direction. An important feature of vector fields of finite type is that the complement of the set of the separatrices of a vector field of finite type is the disjoint union of countably many unbounded connected open sets (named by Markus [10] *canonical regions*) whose boundary has only a finite number of connected components.

We recall that a vector field ξ is *Hamiltonian* when its flow preserves the standard area 2-form $\Omega_0 = dx \wedge dy$, namely when $(\phi^t_\xi)^*\Omega_0 = \Omega_0$ for every t. When $\xi \in \mathfrak{X}_r(\mathbb{R}^2)$ is Hamiltonian, its integral trajectories are the level sets of a regular function H, namely $dH(\xi) = 0$, which justifies the following definition:

Definition 2. *A regular planar vector field ξ (resp. a foliation $\mathcal{F}$) is* intrinsically Hamiltonian *if its integral trajectories (resp. the leaves of $\mathcal{F}$) are the level*

[1] The spaces of leaves are often non-Haussdorf. For an introduction to non-Haussdorf smooth structures, see [8]

sets of a regular function H, i.e. if $dH(\xi)=0$ (resp. $\mathcal{F}=\{dH=0\}$). Moreover, we say that ξ (resp. $\mathcal{F}$) is transversally Hamiltonian *when its integral trajectories (resp. leaves) are transversal at every point to the level sets of a regular function H, i.e. if $dH(\xi)>0$ at every point (resp. if $\mathcal{F}$ and $\{dH=0\}$ are transversal at every point).*

It was proved in [18] (resp. in [4]) that every intrinsically Hamiltonian vector field (resp. every vector field of finite type) in $\mathfrak{X}_r(\mathbb{R}^2)$ is also transversally Hamiltonian.

Lie Derivatives and Main goal. Although in the context of Differential Geometry these two objects are often identified, in order to avoid ambiguities we use here different symbols for a vector field $\xi=(\xi_x,\xi_y)$ and the corresponding Lie derivative operator L_ξ along its flow, namely we set

$$L_\xi f(x_0) = \frac{d}{dt} f(\phi^t_\xi(x_0))\Big|_{t=0} = (\xi_x\partial_x + \xi_y\partial_y)\, f\Big|_{x=x_0}.$$

It is easily seen that $\xi \in \mathfrak{X}_r(\mathbb{R}^2)$ is intrinsically Hamiltonian if and only if the partial differential equation $L_\xi f = 0$ admits a C^1 regular solution and is transversally Hamiltonian if and only if the partial differential inequality $L_\xi f > 0$ has a C^1 solution.

In the present article we focus on the image of the Lie derivative acting on several functional spaces. In particular we consider the restrictions of L_ξ to the space of C^r functions

$$L^{(r)}_\xi : C^r(\mathbb{R}^2) \to C^{r-1}(\mathbb{R}^2)\,,\ r=1,2,\dots,\infty$$

and its extensions

$$L^{(l,p)}_\xi : W^{l,p}_{loc}(\mathbb{R}^2) \to W^{l-1,p}_{loc}(\mathbb{R}^2)\,,\ p\geq 1,\ l=1,2,\dots\,,$$

to the Sobolev spaces $W^{l,p}_{loc}(\mathbb{R}^2)$ of L^p_{loc} functions whose first l *weak* derivatives are also L^p_{loc} (see [1] for a definition of weak derivatives of locally integrable functions). Note that, in order to keep notation light, we will omit the upper index in the operators L_ξ when there is no ambiguity. We endow $C^r(\mathbb{R}^2)$ with the Whitney topology.

Our main goal is studying the images

$$L_\xi\left(C^r(\mathbb{R}^2)\right)\cap C^k(\mathbb{R}^2)\,,\ L_\xi\left(W^{l,p}_{loc}(\mathbb{R}^2)\right)\cap C^k(\mathbb{R}^2).$$

In other words, we study the existence of global C^r or $W^{l,p}_{loc}$ solutions of the cohomological equation when the right hand side is of class C^k. In case there is no regularity loss (i.e. $r=k+1$), the problem reduces to studying the full image $L_\xi(C^{k+1}(\mathbb{R}^2))$.

Notice that, since $L_\xi(C^r(\mathbb{R}^2))\subset C^k(\mathbb{R}^2)$ for $r\geq k+1$, it is enough to consider the cases $r=1,\dots,k+1$. Similarly, since $W^{k+1,p}(\mathbb{R}^2)\subset C^k(\mathbb{R}^2)$ for $p>2$ and $W^{k+2,p}(\mathbb{R}^2)\subset C^k(\mathbb{R}^2)$ for $1\leq p\leq 2$ (e.g. see [6]), it is enough to consider the cases $l=0,\dots,k+1$ for $p>2$ and $l=0,\dots,k+2$ for $1\leq p\leq 2$.

Remark 1. *It makes sense considering the case $l = 0$, namely functions of class $W^{0,p}_{loc} = L^p_{loc}$, because the solutions of (1) are, by definition, C^{k+1} along the integral curves of ξ.*

Example 2. *Consider again the regular vector field $\xi = (2y, 1-y^2)$. As a corollary of Proposition 2 in [4], $1 \notin L_\xi(C^r(\mathbb{R}^2)) \cap C^\infty(\mathbb{R}^2)$ for any r. On the other side $1 \in L_\xi(L^1_{loc}(\mathbb{R}^2)) \bigcap C^\infty(\mathbb{R}^2)$ since, for example, $L_\xi f = 1$ for $f(x,y) = \frac{1}{2}\ln\left|\frac{1+y}{1-y}\right|$. Note that, in particular,*

$$1 \in L_\xi\left(L^1_{loc}(\mathbb{R}^2) \cap C^\infty\left(\mathbb{R}^2 \setminus \{y = \pm 1\}\right)\right) \bigcap C^\infty(\mathbb{R}^2).$$

This behavior is general, in the sense that we can always find weak solutions of (1) that are C^{k+1} outside of the set of separatrices of ξ. Such solutions can be easily constructed via the method of characteristics (e.g. see [2]). Note, finally, that solutions that diverge on just one of the two separatrices can be easily obtained through the L_ξ's weak first-integral $h(x,y) = x + \ln|1-y^2|$.

Main Results. In Section 3 we present some result on the local and global geometry of planar foliations that we will use in Section 4. The section's main result, contained in Theorem 3, is that for such vector fields, locally, the problem of the extension of a solution of the cohomological equation from a saturated neighborhood of a separatrix s_1 to the saturated neighborhood of an adjacent separatrix s_2 can be always reduced to the problem of the extension, from $\mathbb{L}^2_0 = (-\infty, 0] \times \mathbb{R} \setminus \{(0,0)\}$ to the whole $\overline{\mathbb{L}^2_0}$, of a solution of $\partial_y f = g \in C^k(\mathbb{L}^2_0)$. This theorem generalizes Proposition 8 in [4] and fixes a minor mistake in its statement.

In Section 4 we study the images of $L^{(r)}_\xi$ and $L^{(l,p)}_\xi$ Our main results are contained in Theorem 5, where we provide explicit criteria for the solubility of the cohomological equation in the Hamiltonian case, and Theorem 7, were weaker results are provided for the finite-type non-Hamiltonian case. Moreover, in Theorem 6, we show that the solvability of the cohomological equation, in the Hamiltonian case, is stable with respect to small perturbation of the right hand side.

Finally, in Section 5, we present in some detail four concrete examples. In the first two we consider, respectively, the cases of two regular Hamiltonian and non-Hamiltonian vector fields depending only on one variable and with just a pair a separatrices and compare our results with those in [5]. In the last two we consider two case not covered by the results in [5]: the case of a regular Hamiltonian vector field with just a pair a separatrices and not invariant with respect to translations in any direction and the case of a regular Hamiltonian vector field with three separatrices inseparable from each other.

3 Geometry of $\mathcal{F}_\xi$

Both the global and local geometry of $\mathcal{F}_\xi$ play a fundamental role to in the study of the image of $L^{(r)}_\xi$. For example, $L^{(r)}_\xi$ is surjective if and only if $\mathcal{F}_\xi$ has no separatrices (see Theorem 1 in [4]), while explicit conditions for the solvability of (1) can be expressed in terms of $\mathcal{F}_\xi$'s local geometry. Below we prove some global and local geometrical properties of planar foliations that are needed for next section.

We recall that the geometry of the set of separatrices of a planar foliation can be quite non-trivial. In [8] Haefliger and Reeb named *feather* the 1-dimensional simply connected non-Haussdorf manifold obtained by attaching a new line (barb) to each point of a dense countable subset of a given line (stem). By repeating this process recursively on the barbs an infinite number of times, they get a 1-dimensional simply connected non-Haussdorf manifold, which they call *composed feather*, with a dense set of branching (i.e. non-Haussdorf) points. Since every 1-dimensional simply connected non-Haussdorf manifold is the space of leaves of some planar foliation (see [8] for details), this is enough to conclude that there are planar foliations whose separatrices are dense in the plane. Explicit examples are available in literature for C^∞ ([17] and [18]) and C^ω ([13]) foliations of the plane with a set of separatrices dense on some open set. Below we show how to build a new explicit example, simpler and more natural than the ones mentioned above, of C^∞ foliation whose separatrices are dense on the whole plane.

We start by considering the foliation $\mathcal{F}_0$ of the whole plane (x, y) in vertical lines and any C^∞ foliation $\mathcal{S}$ of the vertical half-stripe $S = [-1, 1] \times [0, \infty)$ like the one shown in Fig. 1 (left). The two vertical boundary components $\{\pm 1\} \times [0, \infty)$ of S are leaves of $\mathcal{S}$ and the restriction of $\mathcal{S}$ to some open neighborhood of its third boundary component $[-1, 1] \times \{0\}$ coincides with the restriction to it of $\mathcal{F}_0$. The only two separatrices in $\mathcal{S}$ are the half-line $r = \{-1\} \times [0, \infty)$ and a line s (dark line in Fig. 1) homotopic to $y(1 - x^2) = 1$. We denote by γ any line (dashed line in Fig. 1) intersecting transversally s and cutting all other leaves of $\mathcal{S}$ (except of course r), by t_0 the x-axis, transversal to all leaves of $\mathcal{F}_0$, by P_k the half-plane $y < k$ and by $S_n = [\ell_n, \ell'_n] \times [n, \infty)$, $n = 1, 2, \ldots$, a sequence of vertical half-strips not mutually intersecting. After replacing, on each S_n, the vertical foliation with a suitably rescaled version of $\mathcal{S}$, we get a C^∞ foliation of the plane $\mathcal{F}_1$ coinciding with $\mathcal{F}_0$ in P_1 and whose of separatrices are dense on the open set $U = \pi^{-1}_{\mathcal{F}_1}(\pi_{\mathcal{F}_1}(t)) \supset P_1$.

We denote, respectively, by s_n and γ_n the images, in S_n, of s and γ. Now let t_n be any line transversal to $\mathcal{F}_1$ and coinciding, within S_n, with γ_n, let $U_n = \pi^{-1}_{\mathcal{F}_1}(\pi_{\mathcal{F}_1}(t_n))$ and let $\Phi_n = (\varphi_n, \psi_n) : U_n \to \mathbb{R}^2$ be a rectifying diffeomorphism of the restriction of $\mathcal{F}_1$ to U_n such that: 1. $t_n = \{\psi_n = 0\}$; 2. all leaves are sent to vertical lines and, in particular, $s_n = \{\varphi_n = 0\}$; 3. the leaves of U_n outside S_n lie within $\varphi_n > 0$. With the same construction described above, we can modify each of these foliations in the half-planes $\varphi_n < 0$, $n = 1, 2, \ldots$, to produce a new foliation $\mathcal{F}_2$ whose separatrices are dense in the open set $U \bigcup_{n=1}^{\infty} \pi^{-1}_{\mathcal{F}_2}(\pi_{\mathcal{F}_2}(t_n))$ and coincides with $\mathcal{F}_1$ on P_2.

By repeating this construction recursively, we get a sequence $\{\mathcal{F}_n\}$ of C^∞ foliations of the plane such that $\mathcal{F}_{n+1}|_{P_{n+1}} = \mathcal{F}_n|_{P_{n+1}}$ and the closure of the set of the separatrices of $\mathcal{F}_n$ contains P_n for every $n = 1, 2, \ldots$. Hence the $\mathcal{F}_n$

converge to a smooth foliation $\mathcal{F}_\infty$ with a set of separatrices dense in the whole plane.

Remark 2. *The space of leaves of $\mathcal{F}_\infty$ has the structure of a 1-dimensional, smooth non-Hausdorff smooth manifold with a dense set of* binary *branch points, namely such that, at every branch point, exactly two branches (or* barbs*) meet. It is easy to modify the foliation $\mathcal{S}$ in order to have, at every branch point, the concurrence of any finite number of barbs, or even infinitely (countably) many.*

Remark 3. *A foliation $\mathcal{F}$ of S satisfying the properties we requested above for $\mathcal{S}$ can be built through the C^∞ function $F(x,y) = \arctan \frac{(x^2-1+e^{y-2})(x+1)}{x-1}$. The leaves of $\mathcal{F}$ are the curves $y = 2 + \ln \frac{(1-x)(x^2+2x+1-c)}{x+1}$. Direct calculations show that: 1) as x goes from -1 to $+1$, the leaves cross the strip S from $y=0$ to $y=+\infty$ for $c<0$, while they both come from and go back to $y=+\infty$ for $c>0$; 2) the sets $r_\pm = \{\pm 1\} \times [0,\infty)$ and $s = \{y = 2 + \log(1-x^2)\}$ are leaves of $\mathcal{F}$; 3) r_- and s are the only separatrices of $\mathcal{F}$; 4) $\mathcal{F}$ is Hamiltonian. Through a C^∞ modification of the exponential function, we can modify the foliation so that its leaves are vertical straight lines in some neighborhood of $[-1,1] \times \{0\}$.*

As mentioned in the remark above, $\mathcal{S}$ can be chosen to be Hamiltonian (see Fig. 1, left). Hence, also every $\mathcal{F}_n$, and therefore even $\mathcal{F}_\infty$, is Hamiltonian of the same class. In particular, as a corollary of a Lemma of Weiner [18] stating that the first component projection $\pi : Imm^\infty(\mathbb{R}^2, \mathbb{R}^2) \to Sub^\infty(\mathbb{R}^2, \mathbb{R})$ is surjective, this shows that the topology of immersions of the plane into itself can be quite non-trivial:

Proposition 1. *For every $k = 2, 3, \dots, \infty$ there exist C^∞ immersions $\Phi_{FG} = (F,G) : \mathbb{R}^2 \to \mathbb{R}^2$ such that the foliation $\mathcal{F} = \{dF = 0\}$ is a composed feather where at every branch point meet k barbs.*

At the other end of the spectrum, if in the construction above we use, instead of $\mathcal{S}$, a foliation $\mathcal{S}'$ such that every C^1 function constant on its leaves has gradient equal to zero on the separatrix r (see Fig. 1, right), we end up with a foliation $\mathcal{F}'_\infty$ which cannot be obtained as the level set of any non-constant C^1 function. Indeed, by construction, any C^1 function constant on the leaves of $\mathcal{F}'_\infty$ will have gradient equal to zero on a dense set on the plane, so it will have a gradient identically zero everywhere and therefore it will be constant. Hence $\mathcal{F}'_\infty$ provides a new (and simpler) example of planar foliation without non-trivial first-integrals of class C^1, other than the ones provided by Wazewski [17] and Weiner [18], and an explicit example of foliation having as space of leaves the composed feather defined by Haefliger and Reeb in [8].

At this regard, it is important to notice that, for a foliation in $\mathbb{R}^2$, the lack of a smooth regular first-integral is a relative, rather than absolute, property. Indeed, as a corollary of a result of Kaplan that *every* foliation of the plane has a *continuous* first-integral [9], we can prove the following:

Theorem 1. *Every C^0 foliation of the plane is a C^∞ Hamiltonian foliation for some suitable differential structure.*

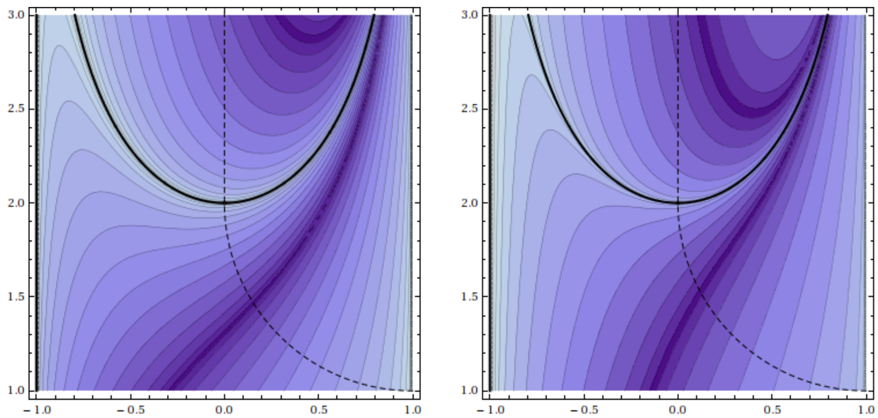

Figure 1: Detail of two C^∞ foliations of the strip $S = [-1,1] \times [0,\infty)$ having, as its only separatrices, the two lines $r = \{-1\} \times [0,\infty)$ and $s = \{y = 2 + \ln(x^2 - 1)\}$. The foliation $\mathcal{S}$, on the left, is Hamiltonian and it consists of the level sets of the C^∞ function $F(x,y) = \arctan \frac{(x^2-1+e^{y-2})(x+1)}{x-1}$. The dashed line γ is a curve cutting s and everywhere transverse to the foliation. The foliation $\mathcal{S}'$, on the right, has the same topology as $\mathcal{S}$ but it is non-Hamiltonian and it consists of the level sets of the (non-regular) C^∞ function $F'(x,y) = \arctan \frac{(x^2-1+e^{y-2})(x+1)^3}{x-1}$. Its two separatrices coincide with the ones of $\mathcal{S}$. Note that $dF' = 0$ on $x = -1$, and so happens for all C^1 functions which are constant on the leaves of $\mathcal{S}'$.

Proof. Let $\mathcal{F}$ be a planar foliation and let $\mathcal{G}$ be any foliation everywhere transversal to it. Assume that $\mathcal{F}$ is not Hamiltonian, otherwise there is nothing to prove, and let F and G two continuous first-integrals of, respectively, $\mathcal{F}$ and $\mathcal{G}$. Then the map $\Phi_{FG} = (F,G) \in C^0(\mathbb{R}^2,\mathbb{R}^2)$ is locally injective. Let now $\mathbb{R}^2{}_{FG}$ be the differential structure on $\mathbb{R}^2$ given by the atlas

$$\mathcal{A}_{\Phi_{FG}} = \{(U_x, (\Phi_{FG})|_{U_x}),\, x \in \mathbb{R}^2\},$$

where U_x is any neighborhood of x such that the restriction of Φ_{FG} to it is injective. In every chart of this atlas, by definition, F and G are represented, respectively, by $F \circ \Phi^{-1}_{FG}$ and $G \circ \Phi^{-1}_{FG}$, namely the projection of the first and second components,and therefore are C^∞ and their differential is nowhere zero. Moreover, by construction, $\mathcal{F} = \{dF = 0\}$ and $\mathcal{G} = \{dG = 0\}$ (note that the differentials of F and G are taken with the respect to the atlas $\mathcal{A}_{\Phi_{FG}}$). Hence, in $\mathbb{R}^2{}_{\Phi_{FG}}$, both $\mathcal{F}$ and $\mathcal{G}$ are C^∞ Hamiltonian foliations. □

Remark 4. *Since, up to dimension 3, every manifold has, modulo diffeomorphisms, only one differentiable structure compatible with its topology (see [12] and [15]), all the differential structures given by the atlases $\mathcal{A}_{\Phi_{FG}}$ are globally diffeomorphic to each other.*

Consider now the set S of all separatrices of $\mathcal{F}_\xi$ inseparable from a fixed separatrix s. On this set it is given a total order relation as follows: $s_1 \geq s_2$ if, given any two transversal t_1, t_2 cutting respectively s_1, s_2, any (and therefore

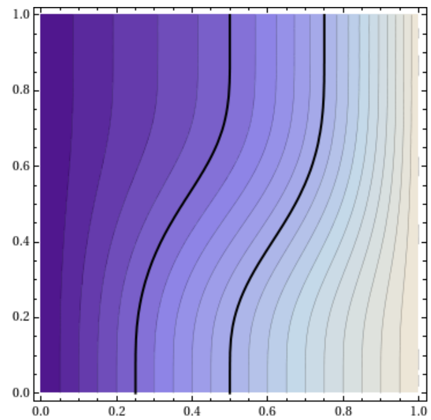

Figure 2: C^∞ foliation of the unit square $[0,1]^2$ coinciding with the vertical foliation in some neighborhood of the horizontal sides $l = [0,1] \times \{0\}$ and $u = [0,1] \times \{1\}$, having the vertical sides as leaves and such that all leaves cutting u at $1/2 \leq x \leq 3/4$ do cut l at $1/4 \leq x \leq 1/2$.

every) integral trajectory of ξ that intersects both t_1 and t_2 cuts t_1 before cutting t_2 (note that integral trajectories cutting both transversal do exist since we are assuming s_1 and s_2 inseparable).

Definition 3. *Given a pair of inseparable leaves s_1, s_2 of a foliation $\mathcal{F}$, we say that they are* adjacent *when, for every other leaf s_3 inseparable from them, holds either $s_3 \geq s_1$ or $s_2 \geq s_3$. We say that a curve γ* separates *two adjacent separatrices s_1 and s_2, or that γ is* between *them, if s_1 and s_2 belong to different connected components of $\mathbb{R}^2 \setminus \gamma$. We say that a foliation $\mathcal{G}$ transversal to $\mathcal{F}$* minimally separates *$\mathcal{F}$ if there is only one leaf of $\mathcal{G}$ between every two adjacent separatrices of $\mathcal{F}$.*

We start with a technical Lemma:

Lemma 1. *Consider the foliation $\mathcal{H}$ in horizontal lines of the set*

$$S = [-1,1] \times [0,1] \setminus [-1/2, 1/2] \times \{0\}.$$

There exists a C^∞ Hamiltonian foliation $\mathcal{T}$ of S such that:

1. *$\mathcal{T}$ is everywhere transversal to $\mathcal{H}$;*
2. *$\mathcal{T}$ separates minimally $\mathcal{H}$;*
3. *the restriction of $\mathcal{T}$ to some neighborhood of the horizontal boundary components of S, $[-1,1] \times \{0,1\}$, is the foliation in vertical lines;*
4. *the vertical boundary components of S, $\{\pm 1\} \times [0,1]$, are leaves of $\mathcal{T}$.*

Proof. The foliation $\mathcal{H}$ has only two separatrices: the lines $s_1 = [-1,-1/2) \times \{0\}$ and $s_2 = (1/2, 1] \times \{0\}$. Let $\mathcal{V}$ be the vertical foliation of S, so that $\mathcal{V}$ is everywhere transversal to $\mathcal{H}$ and s_1 and s_2 are separated by all leaves $\{x = c\} \cap S$ of $\mathcal{V}$ with $-1/2 \le c \le 1/2$.

We start our construction of $\mathcal{T}$ by replacing the foliation in the rectangle $L_1 = [-1, 0] \times [1/2, 1]$ with a suitably rescaled copy of the foliation shown in Fig. 2 and the one in the rectangle $R_1 = [0, 1] \times [1/2, 1]$, specular image of L_1 about the y axis, with a specular image of that foliation. After these substitutions we get a C^∞ foliation $\mathcal{T}_1$ still transversal at every point to $\mathcal{H}$, but where only the leaves crossing $y = 1$ at $-1/4 \le x \le 1/4$ do separate s_1 and s_2.

Now we repeat the procedure recursively by replacing the foliation in the rectangles $L_n = [-2^{2-2n}, 0] \times [1/(n+1), 1/n]$ and $R_n = [0, 2^{2-2n}] \times [1/(n+1), 1/n]$ with suitably rescaled/reflected copies of the foliation shown in Fig. 2. At the step n, therefore, we generate a C^∞ foliation $\mathcal{T}_n$ everywhere transversal to $\mathcal{H}$ and such that only the leaves crossing $y = 1$ at $-1/2^{2n} \le x \le 1/2^{2n}$ do separate s_1 and s_2.

In the limit for $n \to \infty$, we are left with a C^∞ foliation $\mathcal{T}$ everywhere transversal to $\mathcal{H}$ and such that only the leaf $\{0\} \times (0, 1]$ separates s_1 and s_2, i.e. $\mathcal{T}$ separates minimally $\mathcal{H}$. ☐

Theorem 2. *Let $\mathcal{F}$ be a C^0 foliation of $\mathbb{R}^2$. Then there exists a C^0 transverse foliation $\mathcal{G}$ which minimally separates $\mathcal{F}$. Moreover, if $\mathcal{F}$ is C^r with respect to some atlas $\mathcal{A}$ and either Hamiltonian or of finite type, then $\mathcal{G}$ can be chosen to be C^r and Hamiltonian with respect to $\mathcal{A}$.*

Proof. Let $\mathcal{A}$ be any atlas of $\mathbb{R}^2$ where $\mathcal{F}$ is of class C^r. Then, either by Weiner's Lemma in [18] (if $\mathcal{F}$ is Hamiltonian) or by Theorem 2 in [4] (if it is of finite type), there exists a C^r locally injective map $\Phi_{FG} = (F, G)$ (which is an immersion if $\mathcal{F}$ is Hamiltonian) that sends $\mathcal{F}$ and $\mathcal{G}$, respectively, in vertical and horizontal lines.

By definition of inseparable leaves, for every pair of adjacent inseparable leaves $s_1, s_2 \in \mathcal{F}$ cut, respectively, by the transversals $t_1, t_2 \in \mathcal{G}$, the set $U_{12} = \pi_{\mathcal{F}}^{-1}(\pi_{\mathcal{F}}(t_1)) \cap \pi_{\mathcal{F}}^{-1}(\pi_{\mathcal{F}}(t_2))$ contains a saturated one-sided (left or right, in the (F, G) coordinates) neighborhood of s_1 and s_2. Let $s_1 \cup s_2 \subset F^{-1}(a)$, $t_i = G^{-1}(b_i)$, with $b_1 < b_2$, and assume, for the sake of the argument, that $U_{12} \subset F^{-1}((-\infty, a))$. Then $U_{12} \supset R_\epsilon = (a - \epsilon, a) \times (b_1, b_2)$ for some $\epsilon > 0$.

By Lemma 1, we can replace the restriction of $\mathcal{G}$ to R_ϵ with a new foliation in such a way that the new foliation $\mathcal{G}'$ is still Hamiltonian, has the same regularity as $\mathcal{G}$ and separates minimally s_1 and s_2. The proof is concluded by repeating this process for all pairs of adjacent separatrices of $\mathcal{F}$. ☐

The previous result allows us to state a stronger version[2] of Proposition 8 in [4]:

Theorem 3. *Let $\xi \in \mathfrak{X}_r(\mathbb{R}^2)$ be either Hamiltonian or of finite type and let $F \in C^\infty(\mathbb{R}^2)$ be a generator of $\ker L_\xi$ and $G \in C^\infty(\mathbb{R}^2)$ such that $L_\xi G > 0$ and $\mathcal{G} = \{dG = 0\}$ minimally separates $\mathcal{F}_\xi$. Then, for every pair of adjacent separatrices $s_1, s_2 \in \mathcal{F}_\xi$, with $s_1 \cup s_2 \subset F^{-1}(a)$ and separated by $t \in \mathcal{G}$, with $t \subset$*

[2]The present version also fixes a minor mistake in the original Proposition's claim

$G^{-1}(b)$, *and for every leaves* $t_1, t_2 \in \mathcal{G}$, *with* $t_i \subset G^{-1}(b_i)$, *cutting respectively* s_1 *and* s_2, *set* $U = \pi_\xi^{-1}(\pi_\xi(t_1)) \cup \pi_\xi^{-1}(\pi_\xi(t_2))$ *and* $V = \pi_\xi^{-1}(\pi_\xi(t_1)) \cap \pi_\xi^{-1}(\pi_\xi(t_2))$. *The map* $\Phi_{_{FG}} = (F, G) : \mathbb{R}^2 \to \mathbb{R}^2$ *satisfies the following conditions:*

1. *the restriction of* $\Phi_{_{FG}}$ *to* U *is a diffeomorphism onto* $\Phi_{_{FG}}(U)$*;*
2. *the leaves of the restrictions of* $\mathcal{F}_\xi$ *and* $\mathcal{G}$ *to* U *are mapped, respectively, into vertical and horizontal lines;*
3. $\Phi_{_{FG}}(s_i) = \{a\} \times G(s_i)$, *with* $G(s_1) \cup G(s_2) = (b_1', b) \cup (b, b_2')$ *for some* $-\infty \leq b_1' < b$ *and* $b < b_2' \leq \infty$*;*
4. *if* $V \subset F^{-1}((-\infty, a))$ *(resp. if* $V \subset F^{-1}((a, \infty))$*), then* $\Phi_{_{FG}}(t) = (a_1, a) \times \{b\}$ *for some* $-\infty \leq a_1 < a$ *(resp.* $\Phi_{_{FG}}(t) = (a, a_1) \times \{b\}$ *for some* $a < a_1 \leq \infty$*) and* $(a-\epsilon, a) \times (b_1, b_2) \subset \Phi_{_{FG}}(V)$ *(resp.* $(a, a+\epsilon) \times (b_1, b_2) \subset \Phi_{_{FG}}(V)$*) for some* $\epsilon > 0$.

Definition 4. *Under the conditions of the previous theorem, we call* $\Phi_{_{FG}} : U \to \mathbb{R}^2$ *a* normal chart *for the adjacent separatrices* s_1, s_2.

4 $L_\xi(C^r(\mathbb{R}^2)) \cap C^k(\mathbb{R}^2)$ and $L_\xi(W^{l,p}_{loc}(\mathbb{R}^2)) \cap C^k(\mathbb{R}^2)$

It is well known that local solutions to the cohomological equation (1) can be built through the method of characteristics. When $\xi \in \mathfrak{X}_r(\mathbb{R}^2)$ and $g \in C^k(\mathbb{R}^2)$, the only obstruction to the existence of a C^r solution, $0 \leq r \leq k$, is the problem of extending a local solution across pairs of adjacent separatrices (e.g. see [4]). In a normal chart (see Theorem 3), this problem can always be reduced to the following:

Problem 1. *Let* $\mathbb{L}^1 = (-\infty, 0]$, $\mathbb{L}^1_0 = (-\infty, 0)$, $\mathbb{L}^2 = (-\infty, 0] \times \mathbb{R}$ *and* $\mathbb{L}^2_0 = \mathbb{L}^2 \setminus \{(0,0)\}$. *Consider* $g \in C^k(\mathbb{L}^2_0)$ *and* $\varphi \in C^r(\mathbb{L}^1 \times \{-1\})$ *and define on* $\mathbb{L}^1_0 \times \{1\}$ *the function*

$$\psi(x) = \int_{-1}^{1} g(x,t)dt + \varphi(x)$$

(by abuse of notation we use above $\psi(x)$ *for* $\psi(x,1)$ *and* $\varphi(x)$ *for* $\varphi(x,-1)$*). Under which conditions on* g *the function* ψ *can be extended to a* C^r *function at* $x = 0$*? Similarly, assuming* $\varphi \in W^{l,p}_{loc}(\mathbb{L}^1 \times \{-1\}) \cap C^r(\mathbb{L}^1_0 \times \{-1\})$, *under which conditions on* g *is* ψ *of class* $W^{l,p}_{loc}$ *at* $x = 0$*?*

In the study of this problem we will need the following functional spaces. For every $k = 0, 1, 2, \dots$, we denote by $\mathfrak{H}^k(\mathbb{L}^i_0)$, $i = 1, 2$, the ring of left germs at the origin of functions in $C^k(\mathbb{L}^i_0)$, i.e. the equivalence classes determined by the equivalence relation $h \simeq h'$ if h and h' coincide in some left neighborhood of the origin. We focus our attention of the following subrings of $\mathfrak{H}^k(\mathbb{L}^i_0)$: $\mathfrak{G}^{r,k}(\mathbb{L}^i_0)$, with $r = 0, \dots, k$, containing the left germs of all functions belonging to $C^r(\mathbb{L}^i) \cap C^k(\mathbb{L}^i_0)$, and $\mathfrak{W}^{l,p,k}(\mathbb{L}^i_0)$, with $p \geq 1$ and $l = 0, \dots, k+3$, containing the left germs of all functions belonging to $W^{l,p}_{loc}(\mathbb{L}^i) \cap C^k(\mathbb{L}^i_0)$.

Definition 5. *We call* singular left germs at the origin *the elements of the quotient rings* $S\mathfrak{G}^{r,k}(\mathbb{L}_0^i) = \mathfrak{H}^k(\mathbb{L}_0^i)/\mathfrak{G}^{r,k}(\mathbb{L}_0^i)$ *and* $S\mathfrak{W}^{l,p,k}(\mathbb{L}_0^i) = \mathfrak{H}^k(\mathbb{L}_0^i)/\mathfrak{W}^{l,p,k}(\mathbb{L}_0^i)$. *By abuse of notation, we denote by* $S\mathfrak{G}^{k+1,k}(\mathbb{L}_0^2)$ *the singular germs of germs of* C^k *functions which are* C^{k+1} *in the first variable.*

Definition 6. *We denote, respectively, by* $\theta_{r,k} : S\mathfrak{G}^{k,r}(\mathbb{L}_0^2) \to S\mathfrak{G}^{k,r}(\mathbb{L}_0^1)$ *and* $\theta_{l,p,k} : S\mathfrak{W}^{l,p,k}(\mathbb{L}_0^2) \to S\mathfrak{W}^{l,p,k}(\mathbb{L}_0^1)$ *the homomorphisms associating, to the left singular germ at* $(0,0)$ *of a function* $g \in C^k(\mathbb{L}_0^2)$, *the left singular germ at* 0 *of the function* $f(x) = \int_{-1}^{1} g(x,y)dy \in C^k(\mathbb{L}_0^1)$ *modulo, respectively, functions of class* C^r *and* $W^{l,p}_{loc}$ *at* $x = 0$. *Correspondingly, we set* $\Theta_{r,k} = \ker\theta_{r,k}$ *and* $\Theta_{l,p,k} = \ker\theta_{l,p,k}$.

Remark 5. *The homeomorphisms* $\theta_{l,p,k}$ *are well defined because, if* $g \in W^{l,p}_{loc}(\mathbb{L}^2)$, *since, by hypothesis (see Problem 1),* $\varphi \in W^{l,p}_{loc}(\mathbb{L}^1)$, *then*

$$\|\psi\|_{W^{l,p}([-1,0])} \leq \|g\|_{W^{l,p}([-1,0]\times[-1,1])} + \|\varphi\|_{W^{l,p}([-1,0])} < \infty,$$

namely $\psi \in W^{l,p}_{loc}(\mathbb{L}^1)$.

Theorem 4. *The sets* $\Theta_{r,k}$ *and* $\Theta_{l,p,k}$ *satisfy the following properties for all* $k = 0, 1, 2, \dots, \infty$ *and* $p \geq 1$*:*

1. $\Theta_{r,k} \subsetneq \Theta_{r-1,k}$ *for all* $1 \leq r \leq k$*;*
2. $\Theta_{k,k}$ *contains the left singular germs of all* y*-odd* C^k *functions;*
3. $\Theta_{r,k} \subsetneq \Theta_{r,p,k}$ *for all* $0 \leq r \leq k$.

Proof. In [4] we showed that the (left singular germ of the) function

$$g(x,y) = \frac{x}{\sqrt{x^2+y^2}}$$

provides an example of an element belonging to $\Theta_{0,\infty}$ but not to $\Theta_{1,\infty}$. After integrating r times g with respect to x, we get concrete examples of elements belonging to $\Theta_{r,\infty}$ but not to $\Theta_{r+1,\infty}$, which proves point (1). Point (2) is due to the fact that in that case the integral of g in y is zero on every interval symmetric with respect to zero.

Consider now again the function g used to prove point (1). Then $\partial_x g(x,y)$ provides an example of function in $\Theta_{0,1,\infty}$ but not in $\Theta_{0,\infty}$ and Point (3) is then proved by considering, more generally, $g^{1/p}$ and by integrating it with respect to x as in point (1). Similar examples can also be obtained, for example, via the function $g_\alpha(x,y) = \left(x^2+y^2\right)^{-\alpha}$, $\alpha > 0$, which belongs to $\Theta_{l,p,\infty}$ for $0 \leq \alpha \leq \frac{1}{p} - \frac{l}{2}$. For example, $g_\alpha \in \Theta_{0,1,\infty}$ for $0 \leq \alpha \leq 1$ (note that, for $\alpha > \frac{1}{2}$, $g_\alpha \notin \Theta_{0,\infty}$) so that, considering as above $g_\alpha^{1/p}$ and by integrating it with respect to x, we get more examples of functions belonging to $\Theta_{r,p,k}$ but not to $\Theta_{r,k}$. $\square$

As we show below, the sets $\Theta_{r,k}$ and $\Theta_{r,p,k}$ play a fundamental role in the solvability of the cohomological equation.

The Hamiltonian case. When ξ is Hamiltonian with respect to the standard smooth structure, the projection $C^r(\mathcal{F}_\xi) \to C^r(U)$, given by the restriction of a C^r function on $\mathcal{F}_\xi$ to any open set $U \subset \mathcal{F}_\xi$, is surjective (e.g. see [8]), so that the regularity of ψ in Problem 1 only depends on the germ of g at $(0,0)$.

For every pair of adjacent separatrices $s_1, s_2 \in \mathcal{F}_\xi$ separated by $t \in \mathcal{G}$, we denote by a the common value of F at every point of the two separatrices and by b the one of G at every point of t and define the homomorphisms $\theta_{r,k,(a,b)} = \theta_{r,k} \circ T_{(a,b)}$ and $\theta_{l,p,k,(a,b)} = \theta_{l,p,k} \circ T_{(a,b)}$, where $(T_{(a,b)}g)(x,y) = g(x-a, y-b)$.

Theorem 5. *Let ξ be a planar Hamiltonian vector field, F a generator of* $\ker L_\xi$*, G a transversal foliation minimally separating $\mathcal{F}_\xi$ and $\xi'_F = \xi / L_\xi G$ the Hamiltonian vector field of F with respect to the symplectic form $dF \wedge dG$. Then, for all $k = 0, \dots, \infty$, $r = 0, \dots, k$, $l = 0, \dots, k+3$ and $p \geq 1$:*

1. *$g \in L_{\xi'_F}\left(C^r(\mathbb{R}^2)\right) \cap C^k(\mathbb{R}^2)$ iff $[T_{(a_i,b_i)}(\Phi_{FG})_* g]_{S\mathfrak{G}^{r,k}(\mathbb{L}^2_0)} \in \Theta_{r,k}$ for all i;*
2. *$g \in L_{\xi'_F}\left(C^{k+1}(\mathbb{R}^2)\right)$ iff $(\Phi_{FG})_* g$ is C^{k+1} in the first variable and $[T_{(a_i,b_i)}(\Phi_{FG})_* g]_{S\mathfrak{G}^{k+1,k}(\mathbb{L}^2_0)} \in \Theta_{k+1,k+1}$ for all i;*
3. *$g \in L_{\xi'_F}(W^{l,p}_{loc}(\mathbb{R}^2)) \cap C^k(\mathbb{R}^2)$ iff $[T_{(a_i,b_i)}(\Phi_{FG})_* g]_{S\mathfrak{W}^{l,k,p}(\mathbb{L}^2_0)} \in \Theta_{l,p,k}$ for all i.*

Proof. 1. If $g \in L^{(r)}_{\xi'_F}$ then there is a C^r solution f to $L_\xi f = g$. In a normal chart (x', y') of some neighborhood of any pair of adjacent separatrices, the function $\varphi(x') = \int_{-\epsilon}^{\epsilon} g(x', y') dy' : (-\delta, 0) \to \mathbb{R}$ equals $f(x', \epsilon)$ modulo some function belonging to $C^r((-\delta, 0])$, i.e. $[\varphi]_{S\mathfrak{G}^r(\mathbb{L}^1_0)} = 0$. If, on the other side, $[T_{(a_i,b_i)}(\Phi_{FG})_* g]_{S\mathfrak{G}^r(\mathbb{L}^2_0)} \in \Theta_r$ for all pairs of adjacent separatrices s_{1_i}, s_{2_i} with transversals t_{1_i}, t_{2_i}, then we can define any C^r function of one of the transversals and extend the solution to the whole plane with the method of the characteristics. The condition $[T_{(a_i,b_i)}(\Phi_{FG})_* g]_{S\mathfrak{G}^r(\mathbb{L}^2_0)} \in \Theta_r$ grants that on every separatrix we can extend the solution to a C^r solution across the separatrix. The argument works similarly for point *3 mutatis mutandis*.

About point *2*, the argument is the same but we must first prove that the property that $(\Phi_{FG})_* g$ is C^{k+1} in the first variable does not depend on the particular choice of F and G. The reason for this is that every other first-integral F' of ξ only depends on F, so that any other pair (F', G'), where F' is a first-integral and G' a transversal Hamiltonian for $\mathcal{F}_\xi$, is such that $(F', G') = (F'(F), G'(F, G))$. Hence, if $(\Phi_{FG})_* g$ is C^{k+1} in the first argument for one particular choice of F and G, it is so for every other choice. □

Corollary 1. *Under the hypotheses of Theorem 5, let $\mathcal{P}$ be the set of all points (a,b) that separate pairs of adjacent separatrices of $(\Phi_{FG})_*\xi$ in $\Phi_{FG}(\mathbb{R}^2)$. Then the following inclusions hold:*

1. $L_{\xi'_F}(C^r(\mathbb{R}^2)) \cap C^k(\mathbb{R}^2) \supset \Phi^*_{FG}(C^r(\mathbb{R}^2) \cap C^k(\mathbb{R}^2 \setminus \mathcal{P}))$;

2. $L_{\xi'_F}(C^{k+1}(\mathbb{R}^2)) \supset \Phi^*_{FG}(C^{k+1}(\mathbb{R}^2))$;

3. $L_{\xi'_F}(W^{l,p}_{loc}(\mathbb{R}^2)) \cap C^k(\mathbb{R}^2) \supset \Phi^*_{FG}(W^{l,p}_{loc}(\mathbb{R}^2) \cap C^k(\mathbb{R}^2 \setminus \mathcal{P}))$.

Next theorem shows in particular that the solvability of the cohomological equation is stable under small perturbations of its right hand side:

Theorem 6. *Let $\xi \in \mathfrak{X}_r(\mathbb{R}^2)$ be Hamiltonian. Then:*

1. *$L_\xi(C^r(\mathbb{R}^2)) \cap C^k(\mathbb{R}^2)$ is a clopen subset of $C^k(\mathbb{R}^2)$ for all $r = 0, \dots, k$ and $k = 0, \dots, \infty$. In particular, $L_\xi(C^\infty(\mathbb{R}^2))$ is clopen in $C^\infty(\mathbb{R}^2)$;*

2. *$L_\xi(C^{k+1}(\mathbb{R}^2))$, is neither open or closed in $C^k(\mathbb{R}^2)$ for all $k = 0, 1, \dots$;*

3. *$L_\xi(W^{l,p}_{loc}(\mathbb{R}^2)) \cap C^k(\mathbb{R}^2)$ is a clopen subset of $C^k(\mathbb{R}^2)$ for all $l = 0, \dots, k+1$, if $p > 2$, and for all $l = 0, \dots, k+2$, if $1 \leq p \leq 2$, for all $k = 0, 1, \dots$.*

Proof. 1. Set $A = L_\xi(C^r(\mathbb{R}^2)) \cap C^k(\mathbb{R}^2)$ and let $g \in A$. Every positive function $\epsilon \in C^0(\mathbb{R}^2)$ defines a neighborhood U_ϵ of g in the strong C^k topology as the set of all C^k functions g' such that

$$|g'(x,y) - g(x,y)| + \|D_{(x,y)}(g' - g)\| + \cdots + \|D^k_{(x,y)}(g' - g)\| \leq \epsilon(x,y)$$

for every $(x,y) \in \mathbb{R}^2$. If $\eta > 0$ is bounded then, in any normal chart,

$$\lim_{x \to 0^-} \left| \int_{-\eta}^{\eta} \partial^k_x g(x-a, y-b) dy \right| < \infty \text{ iff } \lim_{x \to 0^-} \left| \int_{-\eta}^{\eta} \partial^k_x \left(g(x-a, y-b) + \epsilon(x,y) \right) dy \right| < \infty,$$

where (a,b) are the coordinates of the point that separates the two separatrices in the normal chart. Hence, in all normal charts, $\theta_{r,k} \circ T_{(a,b)}([g]) = \theta_{r,k} \circ T_{(a,b)}([g'])$ for all $g' \in U_\epsilon$, namely $U_\epsilon \subset A$, namely A is open.

Now, let $\{g_n\}$ a sequence of elements of A converging to $g \in C^r(\mathbb{R}^2)$ in the strong topology. Then, almost all the g_n coincide with g outside of some compact set and, therefore, in any normal chart,

$$\lim_{x \to 0^-} \int_{-\epsilon}^{\epsilon} \partial^k_x g(x,y) dy = \lim_{x \to 0^-} \int_{-\epsilon}^{\epsilon} \partial^k_x g_n(x,y) dy,$$

i.e. $\theta_r([g]) = \theta_r([g_n])$, for almost all n, namely A is closed.

2. In case of $L_\xi(C^{k+1}(\mathbb{R}^2))$, it is enough to observe that the property of being C^{k+1} in the first variable in every normal chart is clearly destroyed by a generic C^k small perturbation and is not preserved by C^k convergence unless $k+1 = k$, namely $k = \infty$.

3. The proof is the same as in point 1. The bounds to the values of l are due to the fact that, by the Sobolev embedding theorem, $W^{l,p}(\mathbb{R}^2) \subset C^{k+1}(\mathbb{R}^2)$ for $l > k + 1 + 2/p$ (see [6]) and therefore, for such large values of l, these sets are neither open nor closed in the C^k topology. □

The non-Hamiltonian case. The method we developed for Hamiltonian vector fields is much less powerful when ξ is not Hamiltonian. The main reason for this is that, in this case, the projection $C^r(\mathcal{F}_\xi) \to C^r(U)$ that sends C^r functions f to their restriction $f|_U$ to an open set $U \subset \mathcal{F}_\xi$ is not always surjective when U contains separatrices [8]. Hence there are constraints to the choice of the function φ of Problem 1 since, if φ does not extends to a global C^r first integral of ξ, then the extension of the solution via the method of characteristics will sooner or later diverge on some of the separatrices.

We start by assuming that ξ is of finite type and recall the following property:

Proposition 2. *If $\xi \in \mathfrak{X}_r(\mathbb{R}^2)$ is not Hamiltonian, the differential of any generator of $\ker L^{(r)}_\xi$, $r \geq 1$, is zero on some of the separatrices of ξ. Similarly, the first derivative of every solution of $L^{(r)}_\xi f = g$ on some of the separatrices is determined by g modulo constants.*

Proof. Since ξ is non-Hamiltonian, then the foliation $\mathcal{F}_\xi$ has the following property: there exist two adjacent separatrices s_1 and s_2 such that, taken any two corresponding transversal segments t_1 and t_2, parametrized by the natural parameters η_1, η_2 with respect to the Euclidean metric in such a way that $\eta_i = 0$ is the coordinate of $s_i \cap t_i$ and that both coordinates are positive for the points of t_1 and t_2 inside $\pi^{-1}_{\mathcal{F}_\xi}(\pi^{-1}_{\mathcal{F}_\xi}(t_1) \cap \pi^{-1}_{\mathcal{F}_\xi}(\pi^{-1}_{\mathcal{F}_\xi}(t_2)$, then $\eta_1(\eta_2) = \eta_2^\alpha + O(\eta_2^\beta)$, with $\beta > \alpha$ and $\alpha \neq 1$. In other words, the leaves of $\mathcal{F}_\xi$ approach the two separatrices at different rates. Assume now, for the argument's sake, that $\alpha < 1$, and define a germ of a function $\varphi(\eta_1)$ on t_1. Then, on t_2, this function becomes $\psi(\eta_2) = \varphi(\eta_1(\eta_2)) = \varphi(\eta_2^\alpha + O(\eta_2^\beta))$, so that

$$\left.\frac{d\psi}{d\eta_2}\right|_{\eta_2} = \frac{\alpha}{\eta_2^{1-\alpha}} \left.\frac{d\varphi}{d\eta_1}\right|_{\eta_2^\alpha + O(\eta_2^\beta)} + O(\beta - 1)$$

and therefore we must have $\left.\frac{d\varphi}{d\eta_1}\right|_{\eta_1=0} = 0$ in order to be able to extend φ to a C^1 function beyond t_2.

Regarding the second part, through the method of characteristics we have that

$$\psi(\eta_2) = \int_0^{T(\eta_1(\eta_2),\eta_2)} [(\Phi^t_\xi)^* g](x(\eta_2), y(\eta_2)) dt + \varphi(\eta_1(\eta_2)),$$

where Φ^t_ξ is the flow of ξ. Hence

$$\left.\frac{d\psi}{d\eta_2}\right|_{\eta_2} = \left[\frac{\alpha}{\eta_2^{1-\alpha}} \partial_1 T(\eta_1(\eta_2), \eta_2) + \partial_2 T(\eta_1(\eta_2), \eta_2)\right] \left[(\Phi_\xi^{T(\eta_1(\eta_2),\eta_2)})^* g\right] (x(\eta_2), y(\eta_2)) +$$

$$+ \frac{\alpha}{\eta_2^{1-\alpha}} \left.\frac{d\varphi}{d\eta_1}\right|_{\eta_2^\alpha + O(\eta_2^\beta)} + O(\beta - 1)$$

and therefore we must have

$$\left.\frac{d\varphi}{d\eta_1}\right|_{\eta_1=0} + \lim_{\eta_2 \to 0} \partial_1 T(\eta_1(\eta_2), \eta_2) [(\Phi_\xi^{T(\eta_1(\eta_2),\eta_2)})^* g](x(\eta_2), y(\eta_2)) = 0$$

in order to be able to extend φ to a C^1 function beyond t_2. $\square$

Remark 6. *Note that the derivative of $\psi(\eta_2)$ is not necessarily null at $\eta_2 = 0$.*

Let $F \in \ker L^{(r)}_\xi$ and let $\mathcal{G} = \{dG = 0\}$ be any Hamiltonian transversal foliation which minimally separates $\mathcal{F}_\xi$. Then $\Phi_{FG} = (F, G) : \mathbb{R}^2 \to \mathbb{R}^2$ is a C^r locally injective map whose rank is 1 on some of the separatrices. In order to make F and G both regular, we switch to the differential structure of the plane $\mathbb{R}^2{}_{FG}$. Since both F and G are C^r, then $C^r(\mathbb{R}^2{}_{FG}) \subsetneq C^r(\mathbb{R}^2)$. By repeating all steps as in the previous section, we get the following, weaker, result

Theorem 7. *Let ξ be a planar vector field of finite type, F a generator of $\ker L^{(r)}_\xi$, G a C^r transversal foliation minimally separating $\mathcal{F}_\xi$ and $\xi'_F = \xi / L_\xi G$ the Hamiltonian vector field of F with respect to the symplectic form $dF \wedge dG$. Then, for all $k = 0, \dots, \infty$, $r = 0, \dots, k$, $l = 0, \dots, k+3$ and $p \geq 1$:*

1. $g \in L_{\xi'_F}\left(C^r(\mathbb{R}^2)\right) \cap C^k(\mathbb{R}^2)$ *if* $[T_{(a_i,b_i)}(\Phi_{FG})_* g]_{S\mathfrak{G}^{r,k}(\mathbb{L}^2_0)} \in \Theta_{r,k}$ *for all* i;
2. $g \in L_{\xi'_F}\left(C^k(\mathbb{R}^2)\right)$ *if* $(\Phi_{FG})_* g$ *is* C^{k+1} *in the first variable and* $[T_{(a_i,b_i)}(\Phi_{FG})_* g]_{S\mathfrak{G}^{k+1,k}(\mathbb{L}^2_0)} \in \Theta_{k+1,k+1}$ *for all* i;
3. $g \in L_{\xi'_F}(W^{l,p}_{loc}(\mathbb{R}^2)) \cap C^k(\mathbb{R}^2)$ *if* $[T_{(a_i,b_i)}(\Phi_{FG})_* g]_{S\mathfrak{W}^{l,k,p}(\mathbb{L}^2_0)} \in \Theta_{l,p,k}$ *for all* i.

In this case the conditions are sufficient but not necessary because the cohomological equation can have $C^r(\mathbb{R}^2)$ solutions which do not belong to $C^r(\mathbb{R}^2{}_{FG})$. In $\mathbb{R}^2{}_{FG}$, such solution look like C^r functions whose derivatives of order r diverge on some of the separatrices where $dF = 0$ (see Proposition 2). Nevertheless, Theorem 7 is enough to extend Corollary 1 to vector fields of finite type:

Corollary 2. *Under the hypotheses of Theorem 7, let P be the set of all points (a, b) that separate pairs of adjacent separatrices of $(\Phi_{FG})_*\xi$ in $\Phi_{FG}(\mathbb{R}^2)$. Then the following inclusions hold:*

1. $L_{\xi'_F}(C^r(\mathbb{R}^2)) \cap C^k(\mathbb{R}^2) \supset \Phi^*_{FG}(C^r(\mathbb{R}^2) \cap C^k(\mathbb{R}^2 \setminus P))$;
2. $L_{\xi'_F}(C^{k+1}(\mathbb{R}^2)) \supset \Phi^*_{FG}(C^{k+1}(\mathbb{R}^2))$;
3. $L_{\xi'_F}(W^{l,p}_{loc}(\mathbb{R}^2)) \cap C^k(\mathbb{R}^2) \supset \Phi^*_{FG}(W^{l,p}_{loc}(\mathbb{R}^2) \cap C^k(\mathbb{R}^2 \setminus P))$.

The case when separatrices are not isolated is more pathological. We briefly discuss here only the limit case, when separatrices are dense on the plane in such a way that $C^1(\mathcal{F}_\xi)$ contains only the constant functions. For such a ξ, every $g \in L_\xi(C^r(\mathbb{R}^2))$ determines uniquely, modulo constants, the solution of the cohomological equation (and, therefore, its restriction to any line). In particular, the method of characteristics in this context can be applied only to C^0 functions since, given a transversal t, a generic $\varphi \in C^r(t)$, $r \geq 1$, will lead eventually to some ineliminable divergence. In $\mathbb{R}^2{}_{\Phi_{FG}}$, though, both F ad G are regular

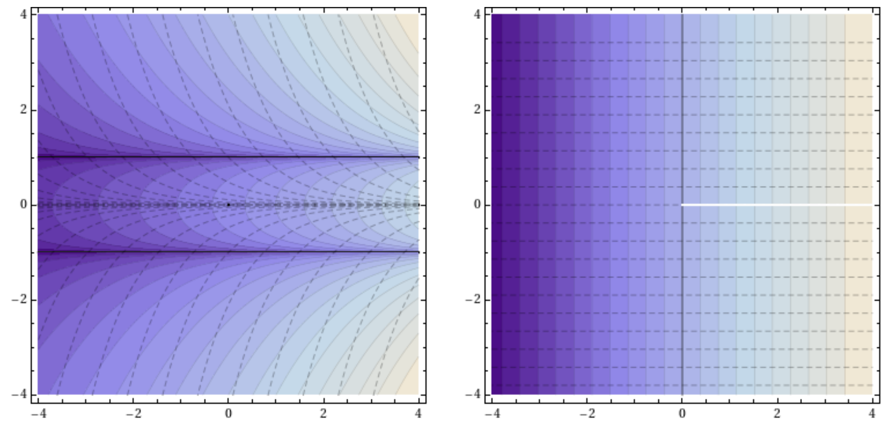

Figure 3: [left] Foliations of the integral trajectories of $\xi = (2y, 1 - y^2)$ (continuous lines) and $\eta = (1, -y)$ (dashed lines). These are tangent, respectively, to the level sets of the regular functions $F(x, y) = (y^2 - 1)e^x$ and $G(x, y) = -2ye^x$. The leaves $y = \pm 1$ are the only pair of inseparable leaves in $\mathcal{F}_\xi$, while $\mathcal{F}_\eta \simeq \mathbb{R}$ has no inseparable leaves. Note that, since ξ is intrinsically Hamiltonian, $C^1(\mathcal{F}_\xi)$ contains regular functions. [right] Image of $\mathcal{F}_\xi$ and $\mathcal{F}_\eta$ via Φ_{FG}. The leaves of $\mathcal{F}_\xi$ become vertical lines, those of $\mathcal{F}_\eta$ horizontal ones. The image, under Φ_{FG}, of the sets $|y| < 1$, $y = 1$, $y = -1$ and $y = 0$ in the left picture are respectively represented, in the right one, by the sets $(-\infty, 0) \times \mathbb{R}$, $\{0\} \times (-\infty, 0)$, $\{0\} \times (0, \infty)$ and $(-\infty, 0) \times \{0\}$.

and the method of characteristic can be used to study the solvability of the cohomological equation in $C^r(\mathbb{R}^2{}_{\Phi_{FG}})$ for all values of r.

Regarding the existence of more regular solutions with respect to the standard differential structure, we recall that, as shown in Proposition 2, g determines the derivative of the restriction of the solution of the equation $L^{(r)}_\xi f = g$ on a dense set A_t of any transversal t.

Definition 7. *We say that a C^r function φ on t is ξ-*compatible *with g if its derivatives coincide with those induced by g, via ξ, in all points of A_t.*

We are lead therefore to the following result:

Theorem 8. *Let $\xi \in \mathfrak{X}_r(\mathbb{R}^2)$ be such that $\dim C^1(\mathcal{F}_\xi) = 1$ and for each $s_i \in \mathcal{S}_\xi$ select a transversal t_i. Then $g \in L_\xi(C^r(\mathbb{R}^2)) \cap C^k(\mathbb{R}^2)$ iff, for each transversal t_i, there exists a C^r function φ_i on t_i ξ-compatible with g.*

5 Examples

In this section we present four model examples.

5.1 $\xi = (2y, 1 - y^2)$

This vector field is Hamiltonian and invariant with respect to horizontal translations. It is easy to see that its only separatrices are the straight lines $y = \pm 1$.

A regular first-integral of L_ξ is the function $F(x,y) = (y^2-1)e^x$ and a solution to the partial differential inequality $L_\xi G \neq 0$ is given by $G(x,y) = -2ye^x$. It is easy to verify that $\mathcal{G} = \{dG = 0\}$ separates minimally $\mathcal{F}_\xi$. Note that, in this particular case, Φ_{FG} is globally injective. In the normal chart $(\mathbb{R}^2, \Phi_{FG})$, the point that separates the two separatrices has coordinates $(0,0)$. A straight calculation shows that

$$\Omega_{FG} = dF \wedge dG = 2(1+y^2)e^{2x} dx \wedge dy$$

and, correspondingly,

$$\xi'_F = \Omega^{-1}_{FG}(dF) = \frac{e^{-x}}{2(1+y^2)}\xi, \;\; \xi'_G = \Omega^{-1}_{FG}(dG) = \frac{e^{-x}}{1+y^2}(1,-y) \; .$$

By Proposition 6 in [4], $(\Phi_{FG})_*\xi_F = \partial_{y'}$ and $(\Phi_{FG})_*\xi_G = \partial_{x'}$, where we set $(x',y') = (F,G)$, and the cohomological equation writes, in the normal chart $(\mathbb{R}^2 \setminus [0,\infty) \times \{0\}, \Phi_{FG})$, as $\partial_{y'}\hat{f} = \hat{g}$.

By Theorem 5, the equation $L_{\xi'_F} f = g \in C^k(\mathbb{R}^2)$ has a C^r solution, $r = 1,\dots,k$, if and only if $[(\Phi_{FG})_* g]_{S\mathfrak{C}^{r,k}(\mathbb{L}^2_0)} \in \Theta_{r,k}$, namely if and only if the C^{k+1} function $\psi(x') = \int_{-1}^{1} \hat{g}(x',y')dy' : [-\infty, 0) \to \mathbb{R}$ can be extended to a C^r function at $x' = 0$. The solution is, instead, $W^{l,p}_{loc}$ if and only if $[(\Phi_{FG})_* g]_{S\mathfrak{W}^{l,p,k}(\mathbb{L}^2_0)} \in \Theta_{l,p,k}$, namely iff $\psi(x')$ has a $W^{l,p}$ singularity at $x' = 0$. Below we discuss in some detail a few concrete cases.

As shown in the proof of Theorem 4, the condition

$$|\hat{g}(x',y')| \leq C\left[(x')^2 + (y')^2\right]^{-\alpha}, \; C > 0$$

is enough to grant the existence of C^0 solutions for $\alpha < 1/2$ and of L^1_{loc} solutions for $1/2 \leq \alpha < 1$.

Consider, for example, the function

$$\hat{g}(x',y') = \left[(x')^2 + (y')^2\right]^{-1/4} \in C^\infty(\mathbb{R}^2 \setminus (0,0)),$$

so that

$$g(x,y) = \Phi^*_{FG}\hat{g}(x,y) = \frac{e^{-x/2}}{\sqrt{1+y^2}} \in C^\infty(\mathbb{R}^2).$$

Then a solution to $L_{\xi'_F} g = f$ is given, in the normal chart (x',y'), by the function

$$\hat{f}(x',y') = \frac{y'}{|x'|^{\frac{1}{2}}}\, {}_2F_1\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{2}; -\frac{(y')^2}{(x')^2}\right),$$

where ${}_2F_1(a,b,c;z)$ is the Gaussian hypergeometric function.
In the (x,y) coordinates therefore the solution writes as

$$f(x,y) = -\frac{2ye^{\frac{x}{2}}}{\sqrt{|1-y^2|}}\, {}_2F_1\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{2}; -\frac{4y^2}{(1-y^2)^2}\right) \in C^0(\mathbb{R}^2)\,,$$

where the continuity is due to the fact that ${}_2F_1\left(\frac{1}{4},\frac{1}{2},\frac{3}{2};-\frac{1}{\epsilon^4}\right)\simeq 2\epsilon$ for $\epsilon\simeq 0$.

On the other side, for

$$\hat g(x',y')=\frac{1}{\sqrt{(x')^2+(y')^2}}\in C^\infty(\mathbb{R}^2\setminus(0,0)),$$

namely

$$g(x,y)=\frac{e^{-x}}{1+y^2}\in C^\infty(\mathbb{R}^2),$$

we get

$$\hat f(x',y')=\ln\left(\sqrt{(x')^2+(y')^2}+y'\right),$$

namely

$$f(x,y)=x+2\ln|1-y|\in L^1_{loc}(\mathbb{R}^2)\cap C^\infty(\mathbb{R}^2\setminus S_\xi).$$

Note that the equation $L_{\xi_F}f(x,y)=\frac{e^{-x}}{1+y^2}$ is equivalent to $L_\xi f(x,y)=2$, which is why we found exactly the solution we already discussed in Example 2.

More generally, the condition

$$|\hat g(x',y')|\leq C\left[(x')^2+(y')^2\right]^{-\alpha},\ \ (x',y')\in U_0,\ C>0$$

where U_0 is some left neighborhood of the origin, writes down, in the original coordinates (x,y), as

$$|g(x,y)|\leq Ce^{2\alpha x}\left[1+y^2\right]^{-2\alpha}\ ,\ \ (x,y)\in S_M$$

where $S_M=(-\infty,-M)\times(-1,1)$, for some $M>0$. Since $1+y^2$ is bounded and larger than 1 for every $y\in S_M$, this means that g will give rise to C^0 solutions of $L_\xi f=g$ when $|g(x,y)|\leq e^{-\alpha x}$ for $\alpha<1/2$ and to L^1_{loc} solutions for $\alpha<1$.

The latter is the same condition given in [5], Proposition 3.1, for the existence of L^1_{loc} solutions to $L_{\xi_1}f=g$. This approach, though, shows that it is enough that this inequality be satisfied by g on some neighborhood of $x=-\infty$ within the strip $|y|<1$ rather than on the whole plane.

5.2 $\xi=(2(2y-1),1-y^2)$

This vector field has the same separatrices as the previous one but it is non-Hamiltonian. A smooth generator of $\ker L_\xi$ is given by $F(x,y)=(y+1)^3(y-1)e^x$ and a regular Hamiltonian function for a transverse foliation that minimally separates $\mathcal{F}_\xi$ is given by $G(x,y)=(2y-1)e^x$. In this case

$$\Omega_{FG}=dF\wedge dG=2(1+y)^2(2-4y+3y^2)e^{2x}dx\wedge dy$$

is degenerate on the separatrix $y=-1$. Correspondingly,

$$\xi'_F=\Omega^{-1}_{FG}(dF)=\frac{e^{-x}}{2(2-4y+3y^2)}\xi$$

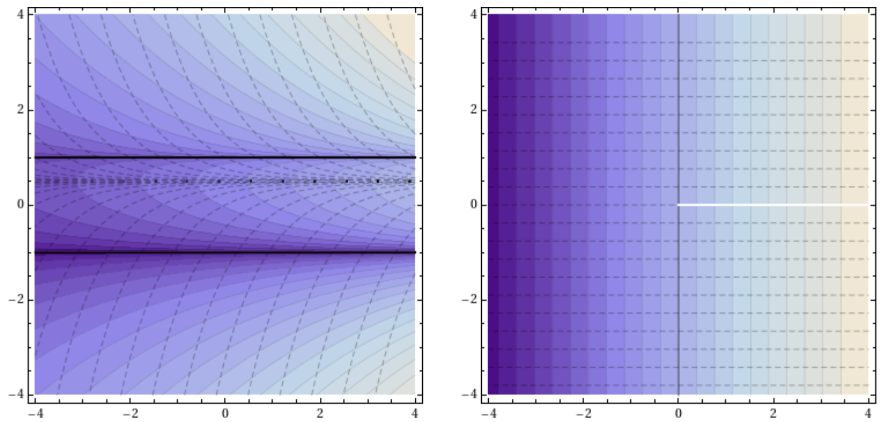

Figure 4: [left] Foliations of the integral trajectories of $\xi = (2(2y-1), 1-y^2)$ (continuous lines) and $\eta = (2, 1-2y)$ (dashed lines). These are tangent, respectively, to the level sets of the function $F(x,y) = (y+1)^3(y-1)e^x$, whose gradient is degenerate on $y=-1$, and of the regular function $G(x,y) = (2y-1)e^x$. The leaf spaces $\mathcal{F}_\xi$ and $\mathcal{F}_\eta$ have the same topology as the corresponding ones in Fig. 3 but this time in $C^1(\mathcal{F}_\xi)$ there are no regular functions, since the derivative of every C^1 function must be zero on the leaf $y=-1$. [right] Image of $\mathcal{F}_\xi$ and $\mathcal{F}_\eta$ via $\Phi_{_{FG}}$. The action of $\Phi_{_{FG}}$ is identical to the one described in Fig. 3, but this time its differential on $y=-1$ is zero, i.e. $\Phi_{_{FG}}$ is not an immersion.

is regular, while

$$\xi'_{_G} = \Omega^{-1}_{_{FG}}(dG) = \frac{e^{-x}}{2(1+y)^2(2-4y+3y^2)}\,(2, 1-2y)\ .$$

diverges on $y=-1$. Similarly, $\Phi_{_{FG}}$ maps, like in the previous example, the whole plane injectively into $\mathbb{R}^2 \setminus [0,\infty)\times\{0\}$, but this time $\Phi_{_{FG}}$ is not an immersion since its differential is zero on the separatrix $y=-1$.

When we restrict $\Phi_{_{FG}}$ to the set $\{|y|<1\}$, the local coordinates $(x',y') = (F,G)$ are well-defined and smooth we can repeat verbatim all calculation for the explicit solutions shown in the previous example. This corresponds to switching differential structure in $\mathbb{R}^2$ and looking for solutions in $\mathbb{R}^2{}_{\Phi_{_{FG}}}$. We recall that, since ξ is of finite type, $C^k(\mathbb{R}^2{}_{\Phi_{_{FG}}}) \subset C^k(\mathbb{R}^2)$; in fact, $C^k(\mathbb{R}^2{}_{\Phi_{_{FG}}})$ is the set of all C^k functions that go to zero as $(y+1)^3$ in a neighborhood of $y=-1$.

Unlike the Hamiltonian case, though, now the solvability conditions based on the germs of g in a neighborhood of the separatrices are only sufficient. New solutions, not covered by the theorems in [5], can be found by letting $\hat g$, in the normal chart coordinates, diverge on the separatrices, as long as $\Phi^*_{_{FG}}\hat g$ has the required differentiability. Consider, for example, the case of

$$\hat g(x',y') = \sqrt[3]{x'}\ \left[y' - \sqrt{(x')^2+(y')^2}\right]^2$$

This function behaves as $4\sqrt[3]{x'}\,(y')^2$, of class C^0, in a neighborhood of the separatrix $\{0\}\times(0,\infty)$, which is the image of $y=-1$, and more regularly, as

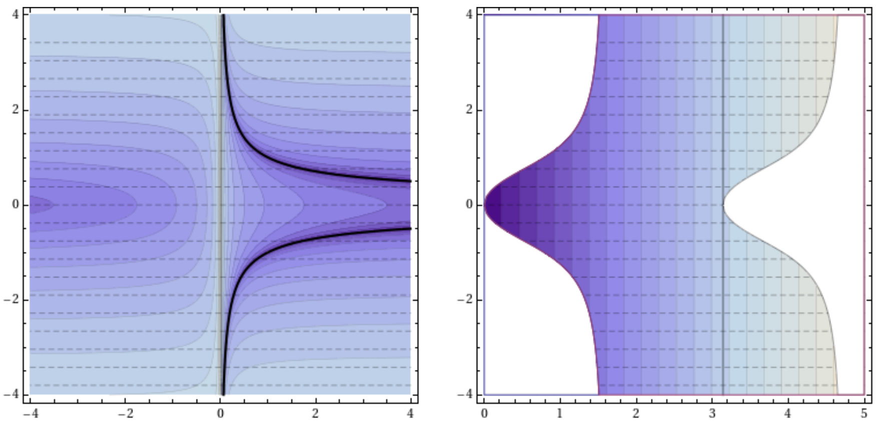

Figure 5: [left] Foliations of the integral trajectories of $\xi = (2x^2y, -1)$ (continuous lines) and $\eta = (1, 0)$ (dashed lines). These are tangent, respectively, to the level sets of the regular functions $F(x, y) = y^2 - e^{-x}$ and $G(x, y) = y$. The leaves $y^2 = e^x$ are the only pair of inseparable leaves in $\mathcal{F}_\xi$, while $\mathcal{F}_\eta \simeq \mathbb{R}$ has no inseparable leaves. Note that, since ξ is intrinsically Hamiltonian, $C^1(\mathcal{F}_\xi)$ contains regular functions. [right] Image of $\mathcal{F}_\xi$ and $\mathcal{F}_\eta$ via Φ_{FG}. The leaves of $\mathcal{F}_\xi$ become vertical lines, those of $\mathcal{F}_\eta$ horizontal ones. The images of the two separatrices are the sets $\{\pi\} \times (-\infty, 0)$ and $\{\pi\} \times (0, \infty)$, the image of the line $y = 0$ is the set $(0, \pi) \times \{0\}$.

$4\sqrt[3]{(x')^{13}}\,(y')^2$, of class C^4, in a neighborhood of the separatrix $\{0\} \times (-\infty, 0)$, which is the image of $y = 1$. In $\mathbb{R}^2{}_{\Phi_{FG}}$, therefore, $\hat{g}$ is of class C^0 and so it gives rise to a globally C^0 solution

$$\hat{f}(x', y') = \sqrt[3]{x'}\left[(x')^2y' + \frac{2}{3}(y')^3 - \frac{2}{3}\left((x')^2 + (y')^2\right)^{3/2}\right]$$

In $\mathbb{R}^2$, instead, the solution is more regular:

$$\left(\Phi^*_{FG}\hat{f}\right)(x, y) = (1 + y)(1 - y)^{1/3}\left[F^2G + \frac{2}{3}G^3 + \frac{2}{3}\left(F^2 + G^2\right)^{3/2}\right]e^{x/3}$$

behaves as $(y - 1)^{13/3}e^{4x/3}$ close to $y = 1$, i.e. of class C^4, and is smooth close to $y = 1$, so we have a globally C^4 solution.

5.3 $\xi = (2x^2y, -1)$

This vector field and the next one are not invariant with respect to any translation and therefore are not covered by the theorems in [5]. We point out that similar examples, with superlinear nonlinearities with respect to x, appear in the study of global Cauchy problems for hyperbolic PDEs on $\mathbb{R}^n$ with initial data on $y = 0$ (e.g. see [7] and [3] and the references therein).

A direct calculation shows that $L_\xi F(x,y) = 0$ for the regular function

$$F(x,y) = \begin{cases} \tan^{-1}\left[y^2 - \frac{1}{x}\right] & , \, x < 0; \\ \pi/2 & , \, x = 0; \\ \tan^{-1}\left[y^2 - \frac{1}{x}\right] + \pi & , \, x > 0, \end{cases}$$

namely $\mathcal{F}_\xi$ is Hamiltonian. The set $F^{-1}(c)$ has two connected components for $c \geq \pi$ and only one for $c < \pi$, so the only inseparable integral trajectories of ξ are the two connected components of $F^{-1}(\pi)$, namely the curves $y = \pm 1/\sqrt{x}$ (see Fig. 5, left). Since the y component of ξ is always non-zero, $\mathcal{F}_\xi$ is everywhere transversal to the foliation in horizontal straight lines and it is minimally separated by it. In particular, $L_\xi G(x,y) > 0$ for $G(x,y) = y$ (see Fig. 5). In this case

$$\Omega_{FG} = dF \wedge dG = \frac{dx \wedge dy}{(1-xy^2)^2 + x^2}.$$

and we get

$$\xi'_F = -((1-xy^2)^2 + x^2)\xi \,, \;\; \xi'_G = ((1-xy^2)^2 + x^2, 0) \,.$$

The image of the plane via the map Φ_{FG} is the set (see Fig. 5, right)

$$\tan^{-1}(y')^2 < x' < \tan^{-1}(y')^2 + \pi.$$

All explicit calculations shown in the first example can be repeated here. For example, this time the condition

$$|\hat{g}(x',y')| \leq C\left[(x')^2 + (y')^2\right]^{-\alpha}, \;\; (x',y') \in U_0,$$

for the existence of regular and weak solutions of the cohomological equation translates, in (x,y) coordinates, into

$$|g(x,y)| \leq C x^{2\alpha}, \;\; x \in S_M,$$

where in this case S_M, $M > 0$, is the portion of the set $y^2 - 1/x < 0$ contained in the half-plane $x > M$. The corresponding condition for solutions of $L_\xi f = g$ is

$$|g(x,y)| \leq C \frac{x^{2\alpha}}{(1-xy^2)^2 + x^2} \leq C' x^{2(\alpha-1)}, \;\; x \in S_M,$$

namely $L_\xi f = g$ admits C^0 solutions when $|g(x,y)| \leq C x^{-1-\epsilon}$ for some $\epsilon > 0$ and L^1_{loc} solutions when $|g(x,y)| \leq C x^{-\epsilon}$ for some $\epsilon > 0$.

5.4 $\xi = (3 - 6e^x(1-y^2) + e^{2x}(19 - 22y^2 + y^4), 3 + 2e^x(5+3y^2) + 3e^{2x}(1-y^2)^2)$

Also this last vector field gives rise to a Hamiltonian foliation. For example, a generator of $\ker L_\xi^{(\infty)}$ is given by the smooth regular function

$$F(x,y) = (1 - (y-2)^2 - e^{-x})(1 - y^2 - e^{-x})(1 - (y+2)^2 - e^{-x}).$$

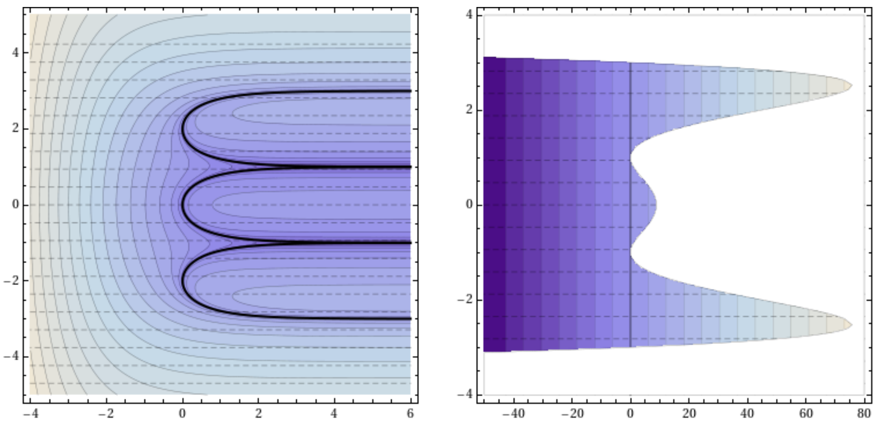

Figure 6: [left] Foliations of the integral trajectories of $\xi = ((3-6e^x(1-y^2)+e^{2x}(19-22y^2+y^4), 3+2e^x(5+3y^2)+3e^{2x}(1-y^2)^2)$ (continuous lines) and $\eta = (1,0)$ (dashed lines). These are tangent, respectively, to the level sets of the regular functions $F(x,y) = (1-(y-2)^2-e^{-x})(1-y^2-e^{-x})(1-(y+2)^2-e^{-x})$ and $G(x,y) = y$. The three leaves $(y \pm 2)^2 = 1 - e^{-x}$ and $y^2 = 1 - e^{-x}$ are the only separatrices of $\mathcal{F}_\xi$ and are all inseparable from each other, while $\mathcal{F}_\eta$ has no inseparable leaves. Note that, since ξ is intrinsically Hamiltonian, $C^1(\mathcal{F}_\xi)$ contains regular functions. [right] Image of $\mathcal{F}_\xi$ and $\mathcal{F}_\eta$ via Φ_{FG}. The leaves of $\mathcal{F}_\xi$ become vertical lines, those of $\mathcal{F}_\eta$ horizontal ones. The image, under Φ_{FG}, of the three separatrices are the sets $\{0\} \times (3,1)$, $\{0\} \times (1,-1)$ and $\{0\} \times (-1,-3)$; the images of the lines $y = \pm 1$ are the sets $(-\infty, 0) \times \{1\}$ and $(-\infty, 0) \times \{-1\}$.

Using the Descartes' rule of signs it is easy to verify that, for $c > 0$, the level sets $F^{-1}(c)$ are connected while, for $c < 0$, each level set consists of three disjoint lines. The three curves in the level set $F^{-1}(0)$ are therefore inseparable from each other. Since the y component of ξ is always different from 0, ξ is transversal to every horizontal line and it is easily seen that the foliation in horizontal lines minimally separates it (see Fig. 6). In particular, $L_\xi G = 1 > 0$ for $G(x,y) = y$. The image of the plane via Φ_{FG} is the set

$$x' < (1-(y'-2)^2)(1-(y')^2)(1-(y'+2)^2)$$

(see Fig. 6, right).

In this case,

$$\Omega_{FG} = dF \wedge dG = dx \wedge dy,$$

so that

$$\xi_F = \Omega_{FG}^{-1}(dF) = \xi\,,\ \ \xi_G = \Omega_{FG}^{-1}(dG) = (1,0)$$

The condition for the existence of solutions of $L_\xi f = g$

$$|\hat{g}(x',y')| \leq C\left[(x')^2 + (y')^2\right]^{-\alpha},\ \ (x',y') \in U_{\pm 1},$$

where U_1 and U_{-1} are left neighborhoods of, respectively, $(0,1)$ and $(0,-1)$, translates now, in (x,y) coordinates, into

$$|g(x,y)| \leq Ce^{3\alpha x},\ \ (x,y) \in S_{M,\pm 1},$$

where in this case $S_{M,1}$ (resp. $S_{M,2}$), $M > 0$, is the portion of the set $F(x, y) < 0$ contained in the half-plane $x > M$. Hence, for example, $L_\xi f = g$ admits C^0 solutions if $|g(x, y)| \leq Ce^{(3/2-\epsilon)x}$, $(x, y) \in S_{M,\pm 1}$, for some $\epsilon > 0$ and L^1_{loc} solutions if $|g(x, y)| \leq Ce^{(3-\epsilon)x}$, $(x, y) \in S_{M,\pm 1}$, for some $\epsilon > 0$.

By modifying suitably this particular $F(x, y)$, it is easy to obtain examples of intrinsically Hamiltonian and intrinsically non-Hamiltonian regular vector fields whose foliation has a single node at which concur any number of inseparable separatrices and which is minimally separated by the horizontal foliation.

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