Hidden structure and uniqueness of solution in a nonlinear system of geometric origin

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Abstract

We discuss the uniqueness of solutions of a family S_n, $n = 2, 3, \dots$, of nonlinear systems of equations in the first orthant of $\mathbb{R}^n$ arising in the context of Kahler geometry. We pose a few conjectures on the general behavior of the system and prove them for low n in the general case and in a particular symmetric case. Some of our proofs are computer-aided since at a certain point we need a large number of calculations. We take into account the computational errors and show that our final results are completely rigorous.

Keywords: Nonlinear analysis; Systems of non-linear equations; Computer-aided proof; Balanced metrics.

AMS codes: 65H10; 53C55.

1 Introduction and motivation

Nonlinear systems of equations over manifolds can be so diverse from each other that there is no hope to develop a general theory for their solvability and uniqueness of solutions and often one has to settle for just numerical results. On the other side, systems arising in applications have often so strong symmetries that sometimes it is, on the contrary possible to attack

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the problem analytically. This is the case of the family S_n of nonlinear systems (see Eq. (2)) that we present in this article.

In order to describe the origin of these systems, we need first to spend some paragraphs to explain the setting in which they arise. In [6], Donaldson presented several numerical results in the setting of Kahler geometry aimed at approximating distinguished metrics, in particular *balanced* ones, following some important theoretical results in [4, 5]. Given a complex manifold X, Donaldson considers an ample holomorphic line bundle $L \to X$, namely a complex 1-dimensional bundle such that, for all k large enough, the (finite dimensional) vector space $H^0(X, L^k)$ of sections of $L^k \to X$ has bases $\{\sigma_1, \dots, \sigma_N\}$ such that the map

$$\Sigma : x \to [\sigma_1(x) : \cdots : \sigma_N(x)]\,,$$

is an embedding of X into $\mathbb{C}P^N$ (for all concepts and definitions of complex differential geometry we use in this section, we refer the reader to any standard text, e.g. see Ch. 2 of [10])

The existence of such embedding is precious because it allows to pull-back onto X the Fubini-Study metric g_{FS} (and so the corresponding form ω_{FS} and measure μ_{FS}) canonically defined on $\mathbb{C}P^N$. In particular, given any basis $\Sigma = \{\sigma_1, \dots, \sigma_N\}$ of $H^0(X, L^k)$, we say that Σ is *balanced* if

$$\langle \sigma_i, \sigma_j \rangle_{FS} = \int_X g_{FS}(\sigma_i(z), \sigma_j(z)) d\mu_{FS}(\Sigma(z)) = c\,\delta_{ij},\ i, j = 1, \dots, N$$

for some constant c.

When $L = K^{\pm m}$ is a power of the canonical bundle K of X, namely K is the bundle of $\dim X$-forms on X (we assume here that K is ample, which is true, for instance, in case of the complex projective spaces $\mathbb{C}P^k$), then we have a canonical measure defined on L in the following way: given any basis $\Sigma = \{\sigma_1, \dots, \sigma_N\}$ of $H^0(X, L)$ orthonormal with respect to the Fubini-Study metric, we define the canonical volume form

$$\mu_{can} = \left[\sum_{i=0}^{N} |\sigma_i|^2\right]^{\pm\frac{1}{m}}$$

and we say that Σ is *canonically balanced* if

$$\langle \sigma_i, \sigma_j \rangle_{can} = \int_X g_{FS}(\sigma_i(z), \sigma_j(z)) d\mu_{can}(\Sigma(z)) = c\,\delta_{ij},\ i, j = 1, \dots, N \qquad (1)$$

for some constant c.

Balanced bases are relevant because they are related to the concept of balanced *metrics*, that play an important role in Kahler geometry. What is now called *balanced geometry* has been object of study in several recent publications (e.g. see [13, 3, 14, 8, 12, 2]; in particular, see [11] for the most updated results on this topic) balanced metrics can be characterized (see [6]) as fixed points of a map T and can be easily gotten numerically this way via iterations. Not much is known, on the contrary, on the corresponding map T_K associated to the canonical case.

The case of interest for the present article is when X is the Riemann sphere $\mathbb{C}P^1$ and $L = K^{-k} = \mathcal{O}(2k)$ is some power of the canonical line bundle $K = \mathcal{O}(-2)$. In this case, it is known that L is the bundle of all homogeneous polynomials in two complex variables z_0 and z_1 of degree $2k$. Modulo multiplication by a scalar, the only balanced basis for $L \to \mathbb{C}P^1$ is $\{\sigma_\ell = \binom{n}{\ell}^{\frac{1}{2}} z_0^{n-\ell} z_1^\ell\}$, $\ell = 0, \dots, n$ (e.g. see Example 2.4 in [1]). In [6], Donaldson's numerical results in case of $\mathbb{C}P^1$ show that iterations of T_K lead to the standard balanced base of $\mathbb{C}P^1$ but no particular reason is provided in that paper. We believe that the reason is that, in this particular case, also the canonically balanced base is unique and coincides with the balanced one.

Let us write explicitly the condition of being canonically balanced in the two simplest cases. First, assume that $L = K^{-1} = \mathcal{O}(2)$, namely $k = 1$. The unique balanced basis of L is $\Sigma = \{z_0^2, \sqrt{2} z_0 z_1, z_1^2\}$. Consider now, for semplicity, the family of bases $\Sigma_a = \{z_0^2, a z_0 z_1, z_1^2\}$, $a \in \mathbb{R}$, and let us write the conditions for Σ_α to be canonically balanced in the projective chart $z_0 = 1$. Then

$$\mu_{can}(\Sigma_a(z)) = \left[\sum_{i=0}^{2} |\sigma_i|^2\right]^{-1} dz \wedge d\bar{z} = \frac{1}{1 + a^2|z|^2 + |z|^4} dz \wedge d\bar{z}$$

and the non-trivial part of condition (1), namely

$$\|\sigma_0\|_{can} = \|\sigma_1\|_{can} = \|\sigma_2\|_{can}$$

becomes, after integration over the angle θ in polar coordinates (r, θ) and switching to $x = r^2$,

$$\int_0^\infty \frac{1}{1 + a^2 x + x^2} dx = \int_0^\infty \frac{a^2 x}{1 + a^2 x + x^2} dx = \int_0^\infty \frac{x^2}{1 + a^2 x + x^2} dx.$$

Note that this is actually a single equation, since the last terms becomes the first after the substitution $y = 1/x$, and is exactly system S_2 with $\alpha = a^2$ (see Eq. (2) and Sec. 3.1).

Assume now that $L = K^{-2} = \mathcal{O}(4)$, namely $k = 2$, so that the unique balanced basis of L is $\Sigma = \{z_0^4, 2z_0^3 z_1, \sqrt{6} z_0^2 z_1^2, 2z_0 z_1^3, z_1^4\}$, and consider the family of bases $\Sigma_{abc} = \{z_0^4, a z_0^3 z_1, b z_0^2 z_1^2, c z_0 z_1^3, z_1^4\}$, $a, b, c \in \mathbb{R}$. The canonical measure is

$$\mu_{can}(z) = \left[\sum_{i=0}^{4} |\sigma_i|^2\right]^{-1/2} dz \wedge d\bar{z} = \frac{1}{(1 + a^2|z|^2 + b|z|^4 + c|z|^6 + |z|^8)^{1/2}} dz \wedge d\bar{z}$$

so that the non-trivial part of condition (1) becomes now

$$\int_0^\infty \frac{1}{p^{1/2}} dx = \int_0^\infty \frac{a^2 x}{p^{1/2}} dx = \int_0^\infty \frac{b^2 x^2}{p^{1/2}} dx = \int_0^\infty \frac{c^2 x^3}{p^{1/2}} dx = \int_0^\infty \frac{x^4}{p^{1/2}} dx,$$

where $p = (1 + a^2 x + b x^2 + c x^3 + x^4)^{1/2}$. As above, the first and last terms are identical and the 3×3 system left is exactly system S_4 with $\alpha_1 = a^2$, $\alpha_2 = b^2$ and $\alpha_3 = c^2$. In general, setting $L = K^{-k}$ results in system S_{2k}. Showing that every S_n has a unique solution would amount to prove the uniqueness of canonically balanced bases for the holomorphic bundles $\mathcal{O}(n) \to \mathbb{C}P^1$.

The article is structured as follows. In Section 2 we pose the main definitions and conjectures and prove some general property. In Section 3 we discuss in detail and prove the conjectures, in particular the uniqueness of the solution, in case of S_2 and S_3. In Section 4 we restrict our attention to the subsystem $\mathcal{S}_n$ symmetric with respect to the symmetry $x \mapsto 1/x$ and prove some stronger property it satisfies. In Section 5 we prove the conjectures for $\mathcal{S}_3$, $\mathcal{S}_4$ and $\mathcal{S}_5$. Finally, in Appendix A we discuss in detail the analysis of errors relative to our compuations, showing how our numerical results lead to rigorous results.

2 Hidden Structure and Conjectures

We denote by $\mathbb{R}^n_+$ the first orthant of $\mathbb{R}^n$, $n \geq 2$, and we consider the family of functions

$$F_{k,n} : \mathbb{R}^n_+ \to \mathbb{R},\ k = 0, \ldots, n,$$

defined by

$$F_{k,n}(\alpha_1,\dots,\alpha_{n-1}) = \int_0^\infty \frac{x^k}{p(x;\alpha_1,\dots,\alpha_{n-1})^{\frac{n+2}{n}}}\,dx,$$

where

$$p(x;\alpha_1,\dots,\alpha_{n-1}) = 1+\alpha_1 x+\dots+\alpha_{n-1}x^{n-1}+x^n.$$

The main object of study of this article is the following system:

$$S_n = \begin{cases} F_{0,n}(\alpha_1,\dots,\alpha_{n-1}) = \alpha_1 F_{1,n}(\alpha_1,\dots,\alpha_{n-1}) \\ \dots \\ F_{0,n}(\alpha_1,\dots,\alpha_{n-1}) = \alpha_{n-1} F_{n-1,n}(\alpha_1,\dots,\alpha_{n-1}). \end{cases} \tag{2}$$

We denote by $\mathcal{Z}_n$ the set of its zeros in the first orthant $\mathcal{O}_n$. Consider the point

$$P_n = \left(\binom{n}{1},\binom{n}{2},\dots,\binom{n}{n-1}\right)$$

and notice that

$$p(x;P_n) = (1+x)^n.$$

A direct calculation shows that

$$F_{k,n}(P_n) = \frac{1}{\binom{n}{k}(n+1)} = \frac{1}{\binom{n}{k}}F_{0,n}(P_n),\ k=0,\dots,n,$$

namely $P_n \in \mathcal{Z}_n$.

Our numerical and analytical explorations lead us to formulate the following conjecture:

Conjecture 1. *For each $n \geq 2$, System (2) has the unique solution P_n in the first orthant, namely $\mathcal{Z}_n = \{P_n\}$.*

The proof of this conjecture for several low-dimensional particular cases is presented in next sections. In the remainder of this section we prove the existence of a hidden non-trivial structure that allows us to reduce this system, at least for the cases discussed in next sections, to a single degenerate equation.

Lemma 2.1. $F_{k,n}(\alpha_1,\dots,\alpha_{n-1}) = F_{n-k,n}(\alpha_{n-1},\dots,\alpha_1)$.

Proof. This is an immediate consequence of the fact that the domain of integration appearing in the F_k is invariant under the transformation $x \mapsto 1/x$. □

The following two functions play a fundamental role in our construction:

$$G_n(\alpha_1, \ldots, \alpha_{n-1}) = \int_0^\infty \frac{1}{p(x; \alpha_1, \ldots, \alpha_{n-1})^{\frac{2}{n}}}\, dx$$

and

$$R_n(\alpha_1, \ldots, \alpha_{n-1}) = 1\frac{\alpha_1}{1} + 2\frac{\alpha_2}{\alpha_1} + \cdots + (n-1)\frac{\alpha_{n-1}}{\alpha_{n-2}} + n\frac{1}{\alpha_{n-1}}.$$

For the sake of conciseness, we will use the notation $A = (\alpha_1, \ldots, \alpha_{n-1})$ and $\widetilde{A} = (\alpha_{n-1}, \ldots, \alpha_1)$ in the rest of the article.

Proposition 2.2. *If* $A = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathcal{Z}_n$, *then*

$$G_n(A) = nF_{0,n}(A) + F_{0,n}(\widetilde{A}).$$

Proof. Let $A \in \mathcal{Z}_n$. We know that

$$F_{0,n} = \alpha_1 F_{1,n} = \cdots = \alpha_{n-1} F_{n-1}.$$

From Lemma 2.1 we also know that, at any point A,

$$F_{n,n}(A) = F_{0,n}(\widetilde{A}).$$

Hence

$$\begin{aligned} G_n(A) &= \int_0^\infty \frac{1}{[1 + \alpha_1 x + \cdots + \alpha_{n-1}x^{n-1} + x^n]^{\frac{2}{n}}}\, dx = \\ &= \int_0^\infty \frac{1 + \alpha_1 x + \cdots + \alpha_{n-1}x^{n-1} + x^n}{[1 + \alpha_1 x + \cdots + \alpha_{n-1}x^{n-1} + x^n]^{\frac{n+2}{2}}}\, dx = \\ &= F_{0,n}(A) + \alpha_1 F_{1,n}(A) + \cdots + \alpha_{n-1}F_{n-1,n}(A) + F_{n,n}(A) = \\ &= nF_{0,n}(A) + F_{0,n}(\widetilde{A}). \end{aligned}$$

□

We use the notation $\widetilde{F}_{0,n}(A) = F_{0,n}(\widetilde{A})$ and similarly for R_n, so that lemma above means that $G_n = nF_{0,n} + \widetilde{F}_{0,n}$ on $\mathcal{Z}_n$.

Proposition 2.3. *The restriction of $F_{0,n}$ to $\mathcal{Z}_n$ is the rational function*

$$\frac{n}{2R_n(\alpha_1,\ldots,\alpha_{n-1})}.$$

Proof. Since

$$p'(x;\alpha_1,\ldots,\alpha_{n-1}) = \alpha_1 + 2\alpha_2 x + \cdots + (n-1)\alpha_{n-1}x^{n-2} + nx^{n-1},$$

then

$$\alpha_1 F_{0,n} + 2\alpha_2 F_{1,n} + \cdots + (n-1)\alpha_{n-1,n}F_{n-2,n} + nF_{n-1,n} =$$
$$= \int_0^\infty \frac{p'(x;\alpha_1,\ldots,\alpha_{n-1})}{[p(x;\alpha_1,\ldots,\alpha_{n-1})]^{\frac{n+2}{n}}}\, dx = -\frac{n/2}{[p(x;\alpha_1,\ldots,\alpha_{n-1})]^{\frac{n}{2}}}\Bigg|_0^\infty = \frac{n}{2}.$$

On $\mathcal{Z}_n$ we have that the left hand side of the identity above is equal to

$$\left(\alpha_1 + 2\frac{\alpha_2}{\alpha_1} + \cdots + (n-1)\frac{\alpha_{n-1}}{\alpha_{n-2}} + n\frac{1}{\alpha_{n-1}}\right) F_{0,n}(A) = R_n(A)F_{0,n}(A),$$

so that finally

$$F_{0,n} = \frac{n}{2R_n}$$

on $\mathcal{Z}_n$. □

Combining Proposition 2.2 with Proposition 2.3, we see that every solution of System (2) must also be a solution of the equation

$$G_n(A) = \frac{n^2}{2R_n(A)} + \frac{n}{2R_n(\widetilde{A})}. \tag{3}$$

Proposition 2.4. *The point P_n is critical for the function*

$$\Gamma_n(A) = G_n(A) - \frac{n^2}{2R_n(A)} - \frac{n}{2R_n(\widetilde{A})}$$

with critical value 0.

Proof. Since the point is symmetric, we can rather consider $G_n - \frac{n(n+1)}{2R}$. First, notice that

$$F_{0,n}(P_n) = \int_0^\infty \frac{1}{((1+x)^n)^{\frac{n+2}{n}}}\,dx = \int_0^\infty \frac{1}{(1+x)^{n+2}}\,dx = \frac{1}{n+1},$$

$$G_n(P_n) = \int_0^\infty \frac{1}{((1+x)^n)^{\frac{2}{n}}}\,dx = \int_0^\infty \frac{1}{(1+x)^2}\,dx = 1$$

and

$$\begin{aligned} R_n(P_n) &= R_n\left(\binom{n}{n-1},\dots,\binom{n}{1}\right) = \\ &= \binom{n}{1} + 2\frac{\binom{n}{2}}{\binom{n}{1}} + \dots + (n-1)\frac{\binom{n}{n-1}}{\binom{n}{n-2}} + n\frac{1}{\binom{n}{n-1}} = \\ &= n + (n-1) + \dots + 2 + 1 = n(n+1)/2, \end{aligned}$$

so that $\Gamma_n(P_n) = 0$.

Moreover,

$$\partial_{\alpha_k} G_n\big|_{P_n} = -\frac{2}{n}\int_0^\infty \frac{x^k}{p(x;\alpha_1,\dots,\alpha_{n-1})^{\frac{n+2}{n}}}\,dx = -\frac{2}{n}F_{k,n} = -\frac{2}{n}\frac{F_{0,n}}{\binom{n}{k}} = -\frac{2}{n(n+1)}\frac{1}{\binom{n}{k}},$$

$$\partial_{\alpha_k} R_n\big|_{P_n} = (n-k)\frac{1}{\binom{n}{n-k-1}} - (n-k+1)\frac{\binom{n}{n-k+1}}{\binom{n}{n-k}^2} = \frac{k+1}{\binom{n}{k}} - \frac{k}{\binom{n}{k}} = \frac{1}{\binom{n}{k}}$$

and

$$\partial_{\alpha_k}\frac{n(n+1)/2}{R_n}\bigg|_{P_n} = -\frac{n(n+1)/2}{R_n^2(P_n)}\partial_{\alpha_k} R_n\big|_{P_n} = -\frac{2}{n(n+1)}\frac{1}{\binom{n}{k}},$$

so that $\partial_{\alpha_k}\Gamma_n\big|_{P_n} = 0$. □

The study of cases $n = 2, 3$ and of the symmetric cases $n = 3, 4, 5, 6$ suggest the following pair of conjectures:

Conjecture 2. *For each $n \geq 2$, Γ_n has a local minimum at P_n. In particular, P_n is an isolated zero of Γ_n.*

Conjecture 3. *For each $n \geq 2$, $\Gamma_n : \mathcal{O}_n \to \mathbb{R}$ has a global unique minimum at P_n. In particular, $\Gamma_n > 0$ in $\mathcal{O}_n \setminus \{P_n\}$.*

Note that Conjecture 3 implies immediately the claim of Conjecture 1.

3 The cases $n = 2$ and $n = 3$

In this section we prove Conjecture 1 for $n = 2$ and $n = 3$. The first case is simple enough that we could prove everything analytically. For the second case we use two different techniques.

First, we use chains of inequalities in order to show that the set of solutions is *bounded* and *away from* the coordinate planes. This shows that the set of solutions is contained in some compact set K. Second, we subdivide K into small cells C_i and expand Γ_n in Taylor series in each C_i (except the cell containing the solution) to verify that $\Gamma_n > 0$ within K. In the cell containing P_n we instead make sure that the determinant of the Hessian of Γ_n is strictly positive. If we get a negative result in any of the cells, we make the subdivision finer and repeat. This algorithm will terminate in finite time if and only if Γ_n is strictly positive away in the complement of P_n.

A full discussion of the methods and error analysis we used in our computations can be found in Appendix A of the present article.

3.1 The case $n = 2$

Here we have a single variable $\alpha = \alpha_1$ and two functions

$$F_{k,2}(\alpha) = \int_0^\infty \frac{x^k}{(1+\alpha x + x^2)^2},\ k = 0, 1.$$

System (2) reduces to the single equation

$$F_{0,2}(\alpha) = \alpha F_{1,2}(\alpha).$$

In this case, therefore, the equivalence of System (2) with Eq. (3) is trivial. The latter writes as

$$\Gamma_2(\alpha) = G_2(\alpha) - \frac{3}{R_2(\alpha)} = 0,$$

where

$$G_2(\alpha) = 2\int_0^1 \frac{dx}{1+\alpha x + x^2} = \begin{cases} \dfrac{\pi - 2\arctan\frac{\alpha}{\sqrt{4-\alpha^2}}}{\sqrt{4-\alpha^2}}, & \alpha \in [0,2) \\ 1, & \alpha = 2 \\ \dfrac{\ln\frac{\alpha+\sqrt{\alpha^2-4}}{\alpha-\sqrt{\alpha^2-4}}}{\sqrt{4-\alpha^2}}, & \alpha \in (2,\infty) \end{cases}$$

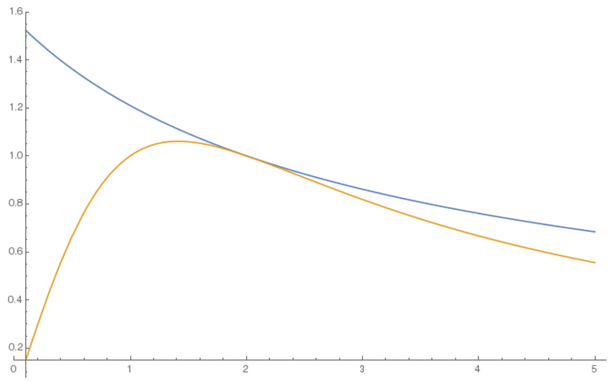

Figure 1: Graphs of G_2 (blue) and $3/R_2$ (beige). The picture suggests that $G_2 \geq 3/R_2$ and that the equal sign only occurs at $\alpha = 2$, where the two graphs are tangent to each other.

and

$$R_2(\alpha) = \alpha + \frac{2}{\alpha}.$$

A direct calculation shows that

$$\left(\sqrt{|4-\alpha^2|}\,\Gamma_2(\alpha)\right)' = 2\sqrt{|4-\alpha^2|}\frac{4-\alpha^2}{(2+\alpha^2)^2},$$

namely $\sqrt{|4-\alpha^2|}\,\Gamma_2(\alpha)$ is monotonic in $(0,2)$ and $(2,\infty)$ and so there are no other zeros of Γ_2 in $(0,\infty)$.

3.2 The case $n = 3$

In this case there are two variables $\alpha = \alpha_1$, $\beta = \alpha_2$ and three functions

$$F_{k,3}(\alpha) = \int_0^\infty \frac{x^k}{(1+\alpha x+\beta x^2+x^3)^{\frac{5}{3}}},\ k = 0,1,2.$$

System (2) is given by

$$\begin{cases} F_{0,3}(\alpha,\beta) = \alpha F_{1,3}(\alpha,\beta) \\ F_{0,3}(\alpha,\beta) = \beta F_{2,3}(\alpha,\beta) \end{cases} \tag{4}$$

and Eq. (3) is

$$G_3(\alpha,\beta) = \frac{9}{2R_3(\alpha,\beta)} + \frac{3}{2R_3(\beta,\alpha)},$$

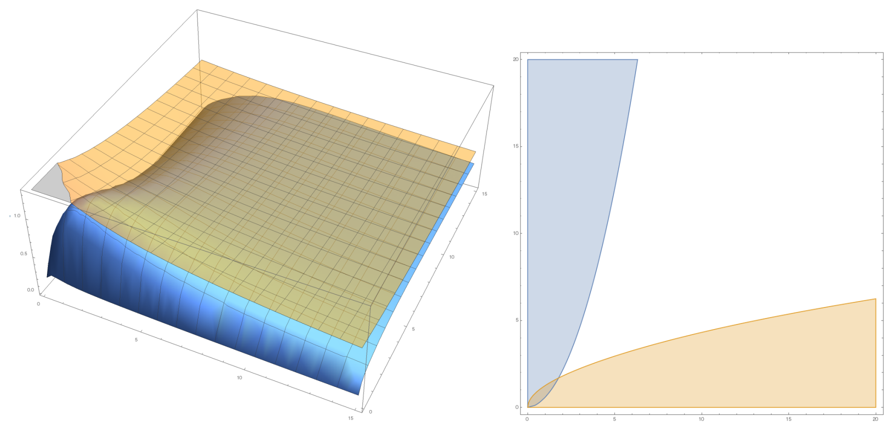

Figure 2: (left) Graphs of G_3 (beige) and $H_3 = 9/(2R_3) + 3/(2\widetilde{R}_3)$ (blue). The picture suggests that $G_3 \geq H_3$ and that the equal sign only occurs at $\alpha = \beta = 3$, where the two graphs are tangent to each other. (right) Graph of the inequalities $\partial_\alpha H_3 > 0$ (blue) and $\partial_\beta H_3 > 0$ (beige).

with
$$G_3(\alpha,\beta) = \int_0^\infty \frac{dx}{(1+\alpha x+\beta x^2+x^3)^{\frac{2}{3}}}$$
and
$$R_3(\alpha,\beta) = \alpha + 2\frac{\beta}{\alpha} + 3\frac{1}{\beta}.$$
The function
$$\Gamma_3(\alpha,\beta) = \int_0^\infty \frac{dx}{(1+\alpha x+\beta x^2+x^3)^{\frac{2}{3}}} - \frac{9/2}{\alpha+2\frac{\beta}{\alpha}+3\frac{1}{\beta}} - \frac{3/2}{\beta+2\frac{\alpha}{\beta}+3\frac{1}{\alpha}}$$
has a local minimum at $(3,3)$. Indeed the Hessian of Γ_3 is
$$H_3 = \begin{pmatrix} \frac{2}{189} & \frac{1}{126} \\ \frac{1}{126} & \frac{2}{189} \end{pmatrix} + \begin{pmatrix} \frac{11}{216} & -\frac{7}{216} \\ -\frac{7}{216} & \frac{5}{216} \end{pmatrix} + \begin{pmatrix} \frac{5}{648} & -\frac{7}{648} \\ -\frac{7}{648} & \frac{11}{648} \end{pmatrix} = \begin{pmatrix} \frac{157}{2268} & -\frac{20}{567} \\ -\frac{20}{567} & \frac{115}{2268} \end{pmatrix}$$
whose eigenvalues are both positive. Hence, $\alpha = \beta = 3$ is an isolated solution of Eqs. (4). Next proposition shows that, similarly to the $n = 2$ case, Γ_3 is non-zero in some neighborhood of the coordinate axes:

Proposition 3.1. $\mathcal{Z}_3 \subset (1,\infty] \times (1,\infty]$

Proof. Set $H_3 = 9/(2R_3) + 3/(2\widetilde{R}_3)$. First of all, notice that $\partial_\alpha R_3 = 1 - 2\beta/\alpha^2$, so that $U_\alpha = \{\partial_\alpha R_3 < 0\}$ is the component containing the positive β semi-axis of the complement of the parabola $2\beta = \alpha^2$; similarly, $\partial_\beta R_3 = 2/\alpha - 3/\beta^2$, so that $U_\beta = \{\partial_\beta R_3 < 0\}$ is the component containing the positive α semi-axis of the complement of the parabola $3\alpha = 2\beta^2$. Notice that, in particular,

$$S = \{(x,x),\ x \in [0,1]\} \subset U_\alpha \cap U_\beta,\ \{1\} \times [1,\infty) \subset U_\alpha,\ [1,\infty) \times \{1\} \subset U_\beta.$$

Set $\tilde{U}_\alpha$ (resp. $\tilde{U}_\beta$) for the symmetric of U_α (resp. U_β) with respect to the diagonal and notice that $U_\alpha \subset \tilde{U}_\beta$ and $\tilde{U}_\alpha \subset U_\beta$. Given that $\partial_\alpha\left(\tilde{R}_3(\alpha,\beta)\right) = \left(\partial_\beta R_3\right)(\beta,\alpha)$ and $\partial_\beta\left(\tilde{R}_3(\alpha,\beta)\right) = \left(\partial_\alpha R_3\right)(\beta,\alpha)$, this means that $\partial_\alpha H_3 > 0$ in U_α and $\partial_\beta H_3 > 0$ in $\tilde{U}_\alpha$.

Denote by S the set $[0,1] \times [0,\infty) \cup [0,\infty) \times [0,1]$. Ultimately, from the facts proved above, it follows that, for every point $P \in S$, either the vertical or the horizontal segment with an endpoint on a coordinate axis and the other on P is entirely contained in S.

All that is left to prove our claim is that $\Gamma_3 > 0$ on ∂S. On the coordinate semiaxis this fact is trivial since $H_3 = 0$ while $G_3 > 0$. Consider now the case $\alpha = 1, \beta \geq 1$. We must prove that

$$\int_0^\infty \frac{dx}{(1+x+\beta x^2+x^3)^{\frac{2}{3}}} - \frac{9/2}{1+2\beta+3/\beta} - \frac{3/2}{\beta+2/\beta+3} > 0.$$

Note that

$$\int_0^\infty \frac{dx}{(1+x+\beta x^2+x^3)^{\frac{2}{3}}} = \int_0^1 \frac{dx}{(1+x+\beta x^2+x^3)^{\frac{2}{3}}} + \int_0^1 \frac{dx}{(1+\beta x+x^2+x^3)^{\frac{2}{3}}} \geq$$

$$\geq 2\int_0^1 \frac{dx}{(1+(2+\beta)x)^{\frac{2}{3}}} = \frac{3}{2+\beta}\left((3+\beta)^{1/3} - 1\right)$$

so it is enough to prove that

$$\frac{6}{2+\beta}\left((3+\beta)^{1/3} - 1\right) > \frac{9/2}{1+2\beta+3/\beta} + \frac{3/2}{\beta+2/\beta+3}.$$

This follows trivially from a direct calculation:

$$3 + \beta - \left[\frac{2+\beta}{4}\left(\frac{3}{1+2\beta+3/\beta} + \frac{1}{\beta+2/\beta+3}\right) + 1\right]^3 =$$

$$\frac{3456+11664\beta+18108\beta^2+17687\beta^3+10150\beta^4+2253\beta^5+80\beta^6+1281\beta^7+2286\beta^8+1643\beta^9+512\beta^{10}}{64(1+\beta)^3(3+\beta+2\beta^2)^3}.$$

Similarly happens for the second half-line. □

With some suitable approximation of the integrals, it is easy to show that all roots of the system are contained in a compact set:

Proposition 3.2. $\mathcal{Z}_3 \subset [1,21]\times[1,21]$

Proof. We need to prove that

$$\int_0^1 \frac{dx}{(1+\alpha x+\beta x^2+x^3)^{\frac{2}{3}}} - \frac{9/2}{\alpha+2\beta/\alpha+3/\beta} +$$

$$+\int_0^1 \frac{dx}{(1+\beta x+\alpha x^2+x^3)^{\frac{2}{3}}} - \frac{3/2}{\beta+2\alpha/\beta+3/\alpha} > 0$$

outside of $[1,21]\times[1,21]$; it is enough indeed to prove that

$$\int_0^1 \frac{dx}{(1+\alpha x+\beta x^2+x^3)^{\frac{2}{3}}} > \frac{9/2}{\alpha+2\beta/\alpha+3/\beta}.$$

Consider first the region $A=\{\alpha\geq 21\}\cap\{\alpha\geq\beta\}$. Inside A

$$\int_0^1 \frac{dx}{(1+\alpha x+\beta x^2+x^3)^{\frac{2}{3}}} \geq \frac{1}{\alpha^{\frac{2}{3}}}\int_0^1 \frac{dx}{(\frac{1}{20}+x+x^2+\frac{1}{20}x^3)^{\frac{2}{3}}} \geq \frac{1.65}{\alpha^{\frac{2}{3}}}.$$

In order to prove that

$$\frac{1.65}{\alpha^{\frac{2}{3}}} > \frac{9/2}{\alpha+2\frac{\beta}{\alpha}+\frac{3}{\beta}}$$

we notice that

$$\left(\frac{2}{9}1.65\left(\alpha+2\frac{\beta}{\alpha}+\frac{3}{\beta}\right)\right)^3 > \frac{\alpha^3}{20}$$

and that $\frac{\alpha^3}{20}\geq\alpha^2$ for $\alpha\geq 20$. Similarly it can be shown that the inequality holds in $B=\{\beta\geq 21\}\cap\{\beta\geq\alpha\}$. □

Finally, we write $[1,21]^2 = O\cup A\cup B$, with $O=[2,3.3]^2$, $A=[1,5]^2\setminus O$ and $B=[1,21]^2\setminus A\cup O$. We subdivide O, A and B in smaller closed cells C_i and in each C_i we expand either Γ_3 to the first order of the Taylor series (if C_i lies within A or B) or the determinant of his Hessian to the zero-th order (if it lies within O) and show that these functions are strictly positive in C_i (see Appendix A).

In Table 1 we report the size of the cells used within the three sets above and the value closer to zero found within them. This shows that there cannot be zeros of Γ_3 in $[1,21]^2$, since within O the function is strictly convex and in $A \cup B$ is strictly positive, proving Conjectures $1,2,3$ for this case.

4 The symmetric case

Calculations in the general case get quickly involved as n grows. We consider in the remainder of this article the simpler particular case when $\alpha_k = \alpha_{n-k}$, namely we restrict the system to the m-dimensional plane of points invariant by the linear transformation $(\alpha_1,\dots,\alpha_n) \mapsto (\alpha_n,\dots,\alpha_1)$, where $m = \lceil n/2 \rceil$. To this category, for $n = 6$, belongs the case considered by Donaldson in [6], coming from a problem in Kahler geometry, that sparked our interest in the general problem.

The restrictions $\mathcal{F}_{k,n} : \mathbb{R}^m_+ \to \mathbb{R}$ of the $F_{k,n}$ write as

$$\mathcal{F}_{k,n}(\alpha_1,\dots,\alpha_m) = \int_0^\infty \frac{x^k}{p(x;\alpha_1,\dots,\alpha_m)^{\frac{n+2}{n}}}\, dx,$$

where

$$p(x;\alpha_1,\dots,\alpha_m) = 1 + \cdots + \alpha_m x^m + \alpha_{m+1} x^{m+1} + \alpha_m x^{m+2} + \cdots + x^n$$

when $n = 2m+1$ and

$$p(x;\alpha_1,\dots,\alpha_m) = 1 + \cdots + \alpha_{m-1} x^{m-1} + \alpha_m x^m + \alpha_m x^{m+1} + \alpha_{m-1} x^{m+2} + \cdots + x^n$$

when $n = 2m$.

Due to Lemma 2.1, $\mathcal{F}_{k,n} = \mathcal{F}_{n-k,n}$ and so the $n \times n$ System (2) reduces to the $m \times m$ system

$$\mathcal{S}_n = \begin{cases} \mathcal{F}_{0,n}(\alpha_1,\dots,\alpha_m) = \alpha_1 \mathcal{F}_{1,n}(\alpha_1,\dots,\alpha_m) \\ \dots \\ \mathcal{F}_{0,n}(\alpha_1,\dots,\alpha_m) = \alpha_m \mathcal{F}_{m,n}(\alpha_1,\dots,\alpha_m) \end{cases} \tag{5}$$

We denote by $\mathcal{Z}_n$ its set of zeros. Note that $\mathcal{P}_n = \left(\binom{n}{1},\dots,\binom{n}{m}\right) \in \mathcal{Z}_n$.

Functions G_n and R_n become

$$\mathcal{G}_n(\alpha_1,\dots,\alpha_m) = \int_0^\infty \frac{dx}{p(x;\alpha_1,\dots,\alpha_m)^{\frac{2}{n}}} = 2\int_0^1 \frac{dx}{p(x;\alpha_1,\dots,\alpha_m)^{\frac{2}{n}}}$$

and

$$\mathcal{R}_n(\alpha_1,\dots,\alpha_m) = \alpha_1+2\frac{\alpha_2}{\alpha_1}+\dots+m\frac{\alpha_m}{\alpha_{m-1}}+(m+1)+(m+2)\frac{\alpha_{m-1}}{\alpha_m}+\dots+(n-1)\frac{\alpha_1}{\alpha_2}+n\frac{1}{\alpha_1}$$

when $n = 2m+1$ and

$$\mathcal{R}_n(\alpha_1,\dots,\alpha_m) = \alpha_1+2\frac{\alpha_2}{\alpha_1}+\dots+m\frac{\alpha_m}{\alpha_{m-1}}+(m+1)\frac{\alpha_{m-1}}{\alpha_m}+\dots+(n-1)\frac{\alpha_1}{\alpha_2}+n\frac{1}{\alpha_1}$$

when $n = 2m$.

Finally, Eq. (3) writes here as

$$\mathcal{G}_n = \frac{n(n+1)}{2\mathcal{R}_n}.$$

We use the notation $\mathscr{G}_n = \mathcal{G}_n - \frac{n(n+1)}{2\mathcal{R}_n}$ and formulate in this setting similar conjectures for the general case:

Conjecture 4. *For each $n \geq 2$, System (5) has the unique solution P_n in the first orthant, namely $\mathcal{Z}_n = \{\mathcal{P}_n\}$.*

Conjecture 5. *For each $n \geq 2$, $\mathscr{G}_n$ has a strict local minimum at $\mathcal{P}_n$ (equivalently, $\mathcal{P}_n$ is an isolated zero of $\mathscr{G}_n$).*

Conjecture 6. *For each $n \geq 2$, $\mathscr{G}_n$ has a global minimum at $\mathcal{P}_n$ (equivalently, $\mathscr{G}_n > 0$ in $\mathcal{O}_m \setminus \{\mathcal{P}_n\}$).*

Proposition 4.1. *The only critical point of $\mathcal{R}_n$ in $\mathcal{O}_m$ is at $\alpha_k = \sqrt{\binom{n}{k}}$, $k = 1,\dots,m$, where it has a global minimum.*

Proof. Consider first the case when n is even. Directly from the definition we get that

$$\partial_{\alpha_m}\mathcal{R}_n = m\frac{1}{\alpha_{m-1}} - (m+1)\frac{\alpha_{m-1}}{\alpha_m} = 0 \tag{6}$$

while

$$\partial_{\alpha_{k-1}}\mathcal{R}_n = (k-1)\frac{1}{\alpha_{k-2}} - k\frac{\alpha_k}{\alpha_{k-1}^2} + (k+1)\frac{1}{\alpha_k} - (k+2)\frac{\alpha_{k-2}}{\alpha_{k-1}^2} = 0,\ k = 2, \dots, m. \tag{7}$$

By multiplying (6) by α_m/α_{m-1} and summing the result to (7), with $k = m$, we get that

$$(m-1)\frac{1}{\alpha_{m-2}} - (m+2)\frac{\alpha_{m-2}}{\alpha_{m-1}^2} = 0.$$

By repeating this process, we ultimately arrive to

$$1 - n\frac{1}{\alpha_1^2} = 0,$$

namely $\alpha_1^2 = n$. By substituting recursively we find that the only critical point of $\mathcal{R}_n$ in $\mathcal{O}_m$ is such that $\alpha_k^2 = \binom{n}{k}$, $k = 1, \dots, m$. Since $\mathcal{R}_n$ diverges at the boundary, this unique critical point must be a minimum. The proof when n is odd is similar. □

Now set $\mathcal{O} = \mathcal{O}_m \setminus [\sqrt{\binom{n}{1}}, \infty) \times \cdots \times [\sqrt{\binom{n}{m}}, \infty)$. The set $\mathcal{O}$ is a neighborhood of the boundary of the first orthant. The lemma below will be used in next section to prove the solution's uniqueness.

Lemma 4.2. *Let $\mathcal{U} = \cup_{i=1}^m U_i$, with $U_i = \{\partial_{\alpha_i}\mathcal{R}_n < 0\}$. Then $\mathcal{U} \supset \mathcal{O}$.*

Proof. Proceeding as in the previous proposition, we see that, for any choice of the other coordinates in the interior of the orthant,

$$\partial_{\alpha_m}\mathcal{R}_n = m\frac{1}{\alpha_{m-1}} - (m+1)\frac{\alpha_{m-1}}{\alpha_m}$$

diverges to $-\infty$ for $\alpha_m \to 0$ and reaches zero in the orthant at the single point $\alpha_m = \binom{n}{m}$.

In case of

$$\partial_{\alpha_{m-1}}\mathcal{R}_n = (m-1)\frac{1}{\alpha_{m-2}} - m\frac{\alpha_m}{\alpha_{m-1}^2} + (m+1)\frac{1}{\alpha_m} - (m+2)\frac{\alpha_{m-2}}{\alpha_{m-1}^2},$$

we have again that the derivative diverges to $-\infty$ for $\alpha_{m-1} \to 0$ for any fixed value of the other coordinates in the interior of the orthant. Moreover, since coordinates other than α_{m-1} appear only in the denominators of the positive

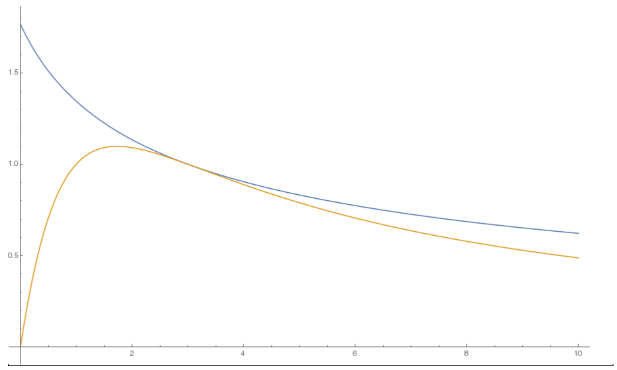

Figure 3: Graphs of $\mathcal{G}_3$ (blue) and $6/\mathcal{R}_3$ (beige). The picture suggests that $\mathcal{G}_3 \geq 6/\mathcal{R}_3$ and that the equal sign only occurs at $\alpha = 3$, where the two graphs are tangent to each other.

terms or in the numerators of the negative ones, we have that $\partial_{\alpha_{m-1}}\mathcal{R}_n < 0$ when $\alpha_{m-1} < \binom{n}{m-1}$ and $\alpha_k > \binom{n}{k}$ for every $k \neq m-1$. The same argument above applies to $\partial_{\alpha_j}\mathcal{R}_n$ for all $j \leq m-1$, so this covers all points of $\mathcal{O}$ except for the parallelotope $\Pi = \prod_i^m (0, \binom{n}{i})$.

Now, take a point $P \in \Pi$ and assume that $\partial_{\alpha_j}\mathcal{R}_n(P) > 0$ for $j = 2, \dots, m$. From case $j = m$ we get that $m/\alpha_{m-1} > (m+1)\alpha_{m-1}/\alpha_m^2$. Using this inequality, from case $j = m-1$ we get that $(m-1)/\alpha_{m-2} > (m+2)\alpha_{m-2}/\alpha_{m-1}^2$ and so on, so that ultimately from case $j = 2$ we get that $2/\alpha_1 > (n-1)\alpha_1/\alpha_2^2$. This means that

$$\partial_{\alpha_1}\mathcal{R}_n = 1 - 2\frac{\alpha_2}{\alpha_1^2} + (n-1)\frac{1}{\alpha_2} - n\frac{1}{\alpha_1^2} < 1 - n\frac{1}{\alpha_1^2} < 0,$$

since $\alpha_1 < n$ in Π, and so that $\Pi \subset \mathcal{U}$. $\square$

5 Proof of the conjectures in the symmetric case for low n

Case $n = 2$ trivially belongs to both the general and the symmetric case, so Section 3.1 represents a first confirmation of the three conjectures above. Below we discuss all cases with $n \leq 5$. The case $n = 6$ will be discussed in a separate article, joint work with A. Loi (U. of Cagliari, Italy), dedicated to the corresponding geometric problem in [6].

5.1 $n = 3$

This case is covered by the proof of the non-symmetric case but here we will provide a fully analytical proof. In this case we have a single variable $\alpha = \alpha_1$ and the functions in play are

$$\mathcal{F}_{k,3}(\alpha) = \int_0^\infty \frac{x^k}{(1+\alpha x+\alpha x^2+x^3)^{\frac{5}{3}}}\,dx,$$

$$\mathcal{G}_3(\alpha) = 2\int_0^1 \frac{dx}{(1+\alpha x+\alpha x^2+x^3)^{\frac{2}{3}}},$$

$$\mathcal{R}_3(\alpha) = \alpha + 2 + 3/\alpha.$$

System (5) is given by the single equation

$$\mathcal{F}_{0,3}(\alpha) = \alpha\mathcal{F}_{1,3}(\alpha) \tag{8}$$

and $\mathscr{G}_3 = 0$ writes explicitly as

$$\int_0^1 \frac{2}{(1+\alpha x+\alpha x^2+x^3)^{\frac{2}{3}}}dx = \frac{6}{\alpha+2+3/\alpha}.$$

We now consider the function

$$C(\alpha) = (\alpha^2+2\alpha+3)\mathscr{G}_3(\alpha),$$

that has the same zeros as $\mathscr{G}_3$. Its second derivative is

$$C''(\alpha) = 4\int_0^1 \frac{1+\rho x+\sigma x^2+\rho x^3+x^4}{(1+x)^{2/3}(1+\alpha x+\alpha x^2+x^3)^{\frac{8}{3}}}dx,$$

where $\rho = \frac{2}{3}(\alpha-5)$ and $\sigma = \frac{2}{9}(23+(2-\alpha)^2)$.

We claim that the integrand in $C''(\alpha)$ is a positive function, and therefore $C'' > 0$. Indeed the denominator of the integrand is always positive and its numerator has all positive coefficients for $\alpha > 5$. Moreover, a direct calculation shows that the numerator has no critical point within the rectangle $R = \{0 \le \alpha \le 5,\ 0 \le x \le 1\}$. At the boundaries, the numerator

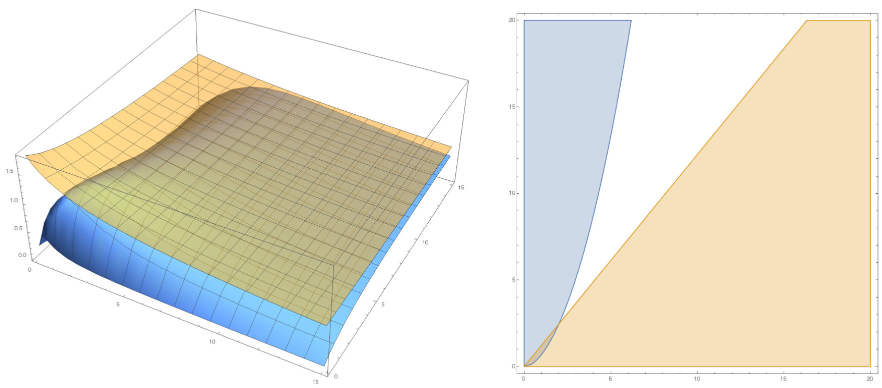

Figure 4: (left) Graphs of $\mathcal{G}_4$ (beige) and $10/\mathcal{R}_4$ (blue). The picture suggests that $\mathcal{G}_4 \geq 10/\mathcal{R}_4$ and that the equal sign only occurs at $\alpha = 4$, $\beta = 6$, where the two graphs are tangent to each other. (right) Graph of the inequalities $\partial_\alpha(10/\mathcal{R}_4) < 0$ (blue) and $\partial_\beta(10/\mathcal{R}_4) < 0$ (beige).

restricts to 9 and $12 + 4\alpha + 2\alpha^2$ on, respectively, $x = 0$ and $x = 1$, and to $9 - 30x + 54x^2 - 30x^3 + 9x^4$ and $9 + 64x^2 + 9x^4$ on, respectively, $\alpha = 0$ and $\alpha = 5$. All four of these polynomials are positive on ∂R and so the minimum of the numerator within R is positive.

Since $C'' > 0$, C is strictly convex and so it can have at most one critical point. Hence, $\mathscr{G}_3$ cannot have other zeros besides $\alpha = 3$.

5.2 $n = 4$

In this case we have two variables, $\alpha = \alpha_1$ and $\beta = \alpha_2$, and four functions

$$\mathcal{F}_{k,n} = \int_0^\infty \frac{x^k}{(1 + \alpha x + \beta x^2 + \alpha x^3 + x^4)^{\frac{3}{2}}},\ k = 0, 1, 2, 3.$$

System (5) is given by

$$\begin{cases} \mathcal{F}_{0,4}(\alpha, \beta) = \alpha \mathcal{F}_{1,4}(\alpha, \beta) \\ \mathcal{F}_{0,4}(\alpha, \beta) = \beta \mathcal{F}_{2,4}(\alpha, \beta) \end{cases} \tag{9}$$

and $\mathscr{G}_4 = 0$ writes explicitly as

$$2 \int_0^1 \frac{dx}{(1 + \alpha x + \beta x^2 + \alpha x^3 + x^4)^{\frac{1}{2}}} = \frac{10}{\alpha + 2\frac{\beta}{\alpha} + 3\frac{\alpha}{\beta} + 4\frac{1}{\alpha}}.$$

Proposition 5.1. *Let $O = \mathbb{R}^2_+ \setminus (1,\infty)^2$. Then $\mathcal{Z}_4 \cap O = \emptyset$.*

Proof. We know from Lemma 5.1 that, close enough to each coordinate axis in the first quadrant, in the direction perpendicular to the axis $\mathcal{G}_4$ decreases while $1/\mathcal{R}_4$ increases.

We claim that $\mathcal{G}_4 > 0$ inside O. All is left to prove is that $\mathcal{G}_4 > 0$ on ∂O. On each coordinate axis, this is due to the fact that $\mathcal{G}_4 > 0$ while $1/\mathcal{R}_4 = 0$. When $\alpha = 1$ and $\beta \geq 1$,

$$\mathcal{G}_4 = 2\int_0^1 \frac{dx}{(1+x+\beta x^2+x^3+x^4)^{\frac{1}{2}}} \geq 2\int_0^1 \frac{dx}{(1+(3+\beta)x)^{\frac{1}{2}}} = 4\frac{\sqrt{4+\beta}-1}{3+\beta}.$$

The fact that

$$4\frac{\sqrt{4+\beta}-1}{3+\beta} > \frac{10}{1+2\beta+3/\beta+4}$$

follows from the observation that

$$4+\beta-\left[1+\frac{5\beta(3+\beta)}{2(2\beta^2+5\beta+3)}\right]^2 = \frac{(3+\beta)p(\beta)}{4(1+\beta)^2(3+2\beta)^2},$$

where $p(\beta) = 36+60\beta-27\beta^2+15\beta^3+16\beta^4$, and that $p > 0$ for $\beta \geq 1$ since, in that range, $p'(\beta) > 0$ and $p(1) > 0$.

Similarly, when $\beta = 1$ and $\alpha \geq 1$,

$$\mathcal{G}_4 = \int_0^1 \frac{2dx}{(1+\alpha x+x^2+\alpha x^3+x^4)^{\frac{1}{2}}} \geq \int_0^1 \frac{2dx}{(1+2(1+\alpha)x)^{\frac{1}{2}}} \geq 2\frac{\sqrt{3+2\alpha}-1}{1+\alpha}.$$

The fact that

$$2\frac{\sqrt{3+2\alpha}-1}{1+\alpha} > \frac{5}{2\alpha+3/\alpha}$$

follows from the observation that

$$3+2\alpha-\left[1+\frac{5(1+\alpha)}{2(2\alpha+3/\alpha)}\right]^2 = \frac{(1+\alpha)q(\alpha)}{4(3+2\alpha^2)^2},$$

where $q(\alpha) = 72-60\alpha+71\alpha^2-65\alpha^3+32\alpha^4$, and that $q > 0$ for $\alpha \geq 1$ since, in that range, $q'(\alpha) > 0$ and $q(1) > 0$. $\square$

Proposition 5.2. $\mathcal{Z}_4 \subset [1,34]\times[1,288]$

Proof. We use the fact that, in $U = \{6\alpha + \beta \geq 200\}$,

$$\int_0^1 \frac{2dx}{(1 + \alpha x + \beta x^2 + \alpha x^3 + x^4)^{\frac{1}{2}}} > \frac{2}{\sqrt{6\alpha + \beta}} \int_0^1 \frac{dx}{(\frac{1}{200} + \frac{1}{6}x + x^2 + \frac{1}{6}x^3 + \frac{1}{200}x^4)^{\frac{1}{2}}}.$$

The integral in the r.h.s. is strictly larger than 2.5, so

$$(\alpha, \beta) \in U \implies \mathcal{G}_4(\alpha, \beta) > \frac{5}{\sqrt{6\alpha + \beta}}.$$

We claim that $5/\sqrt{6\alpha + \beta} > 10/\mathcal{R}_4$ when either $\alpha > 24$ or $\alpha \leq 24, \beta > 288$. Indeed, since $\mathcal{R}_4^2 > \alpha^2 + 4\beta$, for $\alpha > 24$ we have that $\mathcal{R}_4^2/4 > 6\alpha + \beta$. On the other side, since $\mathcal{R}_4^2 > 4\beta^2/\alpha^2 + 4\beta$, for $\alpha \leq 24$ and $\beta > 288$ we have that

$$\mathcal{R}_4^2/4 > \beta^2/24^2 + \beta > 6 \cdot 24 + \beta \geq 6\alpha + \beta$$

when $\beta^2 > 6 \cdot 24^3$, namely when $\beta > 288$. □

We write the rectangle $R = [2, 34] \times [\sqrt{6}, 240]$ as the union of the sets $O = [3.5, 5] \times [5, 7]$, $A = [2, 7] \times [2.4, 9] \setminus O$ and $B = R \setminus A \cup O$. By proceeding exactly as in case of Section 3.2 we verify that $\mathcal{G}_4$ has only one zero in R (see Table 1 for the size of the cells used within the three sets above and the value closer to zero found within them).

5.3 $n = 5$

In this case we still have just two variables, $\alpha = \alpha_1$ and $\beta = \alpha_2$, but now five functions

$$\mathcal{F}_{k,n} = \int_0^\infty \frac{x^k}{(1 + \alpha x + \beta x^2 + \beta x^3 + \alpha x^4 + x^5)^{\frac{7}{5}}},\ k = 0, 1, 2, 3, 4.$$

System (5) is given by

$$\begin{cases} \mathcal{F}_{0,5}(\alpha, \beta) = \alpha \mathcal{F}_{1,5}(\alpha, \beta) \\ \mathcal{F}_{0,5}(\alpha, \beta) = \beta \mathcal{F}_{2,5}(\alpha, \beta) \end{cases} \tag{10}$$

and $\mathcal{G}_4 = 0$ writes explicitly as

$$2\int_0^1 \frac{dx}{(1 + \alpha x + \beta x^2 + \beta x^3 + \alpha x^4 + x^5)^{\frac{2}{5}}} = \frac{15}{\alpha + 2\frac{\beta}{\alpha} + 3 + 4\frac{\alpha}{\beta} + 5\frac{1}{\alpha}}.$$

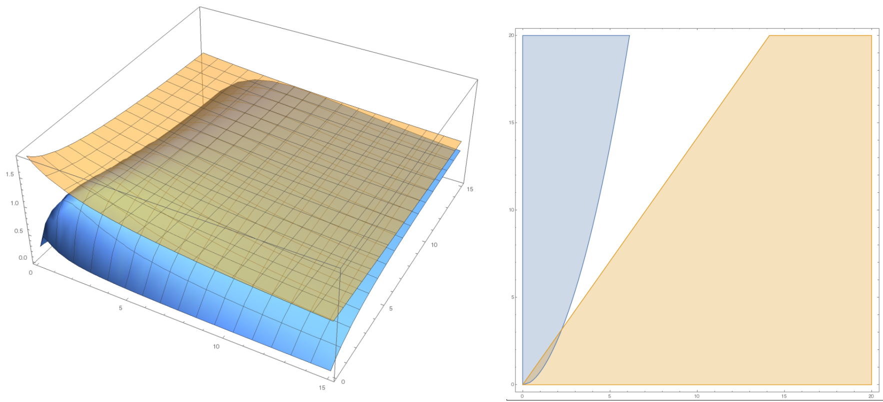

Figure 5: (left) Graphs of $\mathcal{G}_5$ (beige) and $15/\mathcal{R}_5$ (blue). The picture suggests that $\mathcal{G}_5 \geq 15/\mathcal{R}_5$ and that the equal sign only occurs at $\alpha = 5$, $\beta = 10$, where the two graphs are tangent to each other. (right) Graph of the inequalities $\partial_\alpha(15/\mathcal{R}_5) < 0$ (blue) and $\partial_\beta(15/\mathcal{R}_5) < 0$ (beige).

Proposition 5.3. *Let $O = \mathbb{R}^2_+ \setminus (1,\infty)^2$. Then $\mathcal{Z}_5 \cap O = \emptyset$.*

Proof. As in Prop 5.1, all is left to prove is that $\mathscr{G}_5 > 0$ on ∂O. On each coordinate axis, this is due to the fact that $\mathcal{G}_5 > 0$ while $1/\mathcal{R}_5 = 0$. When $\alpha = 1$ and $\beta \geq 1$,

$$\mathcal{G}_5 = \int_0^1 \frac{2dx}{(1 + x + \beta x^2 + \beta x^3 + x^4 + x^5)^{\frac{2}{5}}} \geq$$

$$\geq \int_0^1 \frac{2dx}{[1 + (3+2\beta)x]^{\frac{2}{5}}} = \frac{10}{3}\frac{(4+2\beta)^{3/5} - 1}{3+2\beta}.$$

The fact that

$$\frac{10}{3}\frac{(4+2\beta)^{3/5} - 1}{3+2\beta} > \frac{15}{1 + 2\beta + 3 + 4/\beta + 5}$$

follows from the observation that

$$(4+2\beta)^3 - \left[1 + \frac{9}{2}\,\frac{3+2\beta}{1 + 2\beta + 3 + 4/\beta + 5}\right]^5 = \frac{(3+2\beta)r(\beta)}{32(4+\beta)^5(1+2\beta)^5}$$

where $r(\beta)$ is a polynomial with all positive coefficients.

Similarly, when $\beta = 1$ and $\alpha \geq 1$,

$$\mathcal{G}_5 = \int_0^1 \frac{2dx}{(1+\alpha x + x^2 + x^3 + \alpha x^4 + x^5)^{\frac{2}{5}}} \geq$$

$$\geq \int_0^1 \frac{2dx}{[1+(3+2\alpha)x]^{\frac{2}{5}}} = \frac{10}{3}\frac{(4+2\alpha)^{3/5}-1}{3+2\alpha}.$$

The fact that

$$\frac{10}{3}\frac{(4+2\alpha)^{3/5}-1}{3+2\alpha} > \frac{15}{\alpha+2/\alpha+3+4\alpha+5/\alpha}$$

follows from the observation that

$$(4+2\alpha)^3 - \left[1 + \frac{9}{2}\frac{3+2\alpha}{\alpha+2/\alpha+3+4\alpha+5/\alpha}\right]^5 = \frac{(3+2\alpha)s(\alpha)}{32(7+3\alpha+5\alpha^2)^5}$$

where $s(\alpha)$ is a polynomial with all positive coefficients. □

Let us show that $\mathcal{Z}_5$ is bounded.

Proposition 5.4. $\mathcal{Z}_5 \subset [1,32] \times [1,400]$

Proof. We use the fact that, in $U = 6\alpha + \beta \geq 300$,

$$2\int_0^1 \frac{dx}{(1+\alpha x+\beta x^2+\beta x^3+\alpha x^4+x^5)^{\frac{2}{5}}} =$$

$$= \frac{2}{(6\alpha+\beta)^{\frac{2}{5}}}\int_0^1 \frac{dx}{(\frac{1}{300}+\frac{1}{6}x+x^2+x^3+\frac{1}{6}x^4+\frac{1}{300}x^5)^{\frac{2}{5}}} >$$

$$> \frac{3.91}{(6\alpha+\beta)^{\frac{2}{5}}}.$$

Now,

$$\frac{3.91}{(6\alpha+\beta)^{\frac{2}{5}}} > \frac{15}{\mathcal{R}_5} \implies (6\alpha+\beta)^2 < k\mathcal{R}_5^5,$$

where $k = 0.0012$. Since $\mathcal{R}_5^5 > \alpha^5 + 10\alpha^3\beta + 40\alpha\beta^2$, for $\alpha \geq 32$ we have that $k\alpha^3 \geq 36$, $k10\alpha^2 \geq 12$ and $40k\alpha > 1$, namely

$$k\mathcal{R}_5^5 > (6\alpha + \beta)^2.$$

Since we have also that $\mathcal{R}_5^5 > 80\beta^3/\alpha + 360\beta^2 + 480\beta^3/\alpha^2 + 80\beta^4/\alpha^3$ and, for $\alpha \leq 32$ and $\beta > 400$, we have the inequalities $16k\beta^3/\alpha > 36\alpha^2$, $64k\beta^2/\alpha > 12\alpha$ and $k(360\beta^2 + 480\beta^3/\alpha^2 + 80\beta^4/\alpha^3) > 1$, even in this range we have that

$$k\mathcal{R}_5^5 > (6\alpha + \beta)^2.$$

□

We write the rectangle $R = [\sqrt{5}, 50] \times [\sqrt{10}, 401]$ as the union of the sets $O = [4,6] \times [9,11]$, $A = [2.2, 11] \times [3, 13] \setminus O$ and $B = R \setminus A \cup O$. By proceeding exactly as in case of Sections 3.2 and 5.2 we verify that $\mathcal{G}_5$ has only one zero in R (see Table 1 for the size of the cells used within the three sets above and the value closer to zero found within them).

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Appendix A Computations

The positivity of a smooth function f on a compact set K can be verified directly, in principle, by considering a finite set of small cells P_i covering K and expanding f in its Taylor series f_i up to some finite order k_i in each of the P_i. If, by making the P_i small enough (and therefore their number large enough), the amplitudes of Taylor's error terms τ_i on each P_i becomes smaller than the minimum value of f_i on P_i, this will grant the positivity of f over the each P_i and therefore on the whole K.

Notice that the number of points needed to make the errors small enough depend on the behavior of the derivatives of order $k_i + 1$, appearing in the τ_i terms and, depending on the function, this approach could easily need way too many points for this calculation to be done by hand in a reasonable time even on a powerful computational cluster. Nevertheless for many simple functions, including the one we deal with in the present article, this number is reasonable and can be done on a single computer in a reasonable time (roughly a few tens of hours).

Notice also that, while real-numbers calculations made on nowadays computers – namely floating points calculations – in general involve inherently computational errors and therefore in general cannot be used to verify exactly *equalities*, there is no such problem for *inequalities*, namely numerical calculations in floating point format can be used to prove *exactly* whether some inequality holds or not provided it is possible to make sure that the error term is small enough. Luckily, this is exactly our case.

Below we illustrate our computational procedure in the particular symmetric case $n = 4$. We denote the absolute and relative floating point errors in the evaluation of some function F by, respectively, δ_F and ρ_F. Let us consider first $\mathcal{R}_4$ and its derivatives. The evaluation of $\mathcal{R}_4$ requires 3 multiplications, 3 divisions and 3 additions. The multiplications are all of numbers exactly representable in floating point format and, whithin the range allowed to α and β, their product are numbers within the double precision allowed range, so they are exact too. In the 3 divisions the relative errors sum up but, since the numbers involved are exactly represented, the absolute error in each operation is of 1 unit on the last significant digit. The largest summand of α and β within the rectangle $R = [2, 34] \times [\sqrt{6}, 240]$ is 240, so that $4 + 2\sqrt{6} \le \mathcal{R}_4 < 960$ and the absolute and relative errors are $\delta_{\mathcal{R}_4} < 4 \cdot 240 \cdot 2^{-53} \lesssim 10^{-13}$ and $\rho_{\mathcal{R}_4} < \delta_{\mathcal{R}_4}/8.8 \lesssim 1.2 \cdot 10^{-14}$. Given that the numerator is exactly representable in floating point format, the same bound applies to the relative error of $10/\mathcal{R}_4$.

In case of the first and second partial derivatives of $10/\mathcal{R}_4$ the denominator will be, respectively, $\mathcal{R}_4^2$ and $\mathcal{R}_4^3$ while the numerator will contain, respectively, a term $\partial_\mu \mathcal{R}_4$ or a combination $2\partial_\mu \mathcal{R}_4 \partial_\nu \mathcal{R}_4 - \partial^2_{\mu\nu} \mathcal{R}_4$. Given that all second derivatives are negative in the whole first octant while, by contruction, in $\mathcal{O}$ all first derivatives are positive, no phenomenon of catastrophic cancellation can occur. A direct check shows that $|\partial_\mu \mathcal{R}_4| \le \mathcal{R}_4/\mu$ for both $\mu = \alpha, \beta$, so that the first derivatives of $\mathcal{R}_4$ are bound by the same relative error of $\mathcal{R}_4$ itself while $\partial_\mu(10/\mathcal{R}_4)$ has relative error double than it, namely $\rho_{\partial_\mu \mathcal{R}_4} \le 2.4 \cdot 10^{-14}$. In case of the second derivatives we have the two summands $2\partial_\mu \mathcal{R}_4 \partial_\nu \mathcal{R}_4 / \mathcal{R}_4^3 \le 2/(\mu\nu\mathcal{R}_4)$ and $10(\partial^2_{\mu\nu}\mathcal{R}_4)/\mathcal{R}_4^2 \le 10/(\mu\nu\mathcal{R}_4)$. The terms $\partial^2_{\mu\nu}\mathcal{R}_4$ are all sums of either one or two terms which, inside

R, are all bounded from above by 180. The absolute errors of the first and second summands are therefore bounded from above by, respectively, $7 \cdot 1.2 \cdot 10^{-13} \cdot \frac{2}{4(4+\sqrt{6})} \lesssim 5 \cdot 10^{-14}$ and $4 \cdot 1.2 \cdot 10^{-13} \cdot \frac{3600}{(4+\sqrt{6})^2} \lesssim 4 \cdot 10^{-11}$. Hence $\delta_{\partial^2_{\mu\nu}\mathcal{R}_4} \leq 4 \cdot 10^{-11}$ and $\rho_{\partial^2_{\mu\nu}\mathcal{R}_4} \leq 4 \cdot 10^{-11} \cdot \frac{12}{4(4+\sqrt{6})} \lesssim 2 \cdot 10^{-11}$. We'll see below that, due to the much larger errors in the evaluation of the integrals in $\mathcal{G}_4$, all these errors can be ignored.

Let us now discuss the error on $\mathcal{G}_4$ and its derivatives. First, we notice that the integral defining $\mathcal{G}_4$ can be suitably re-written as an integral over a compact interval, since

$$\int_0^\infty \frac{1}{\sqrt{p}}\, dx = 2\int_0^1 \frac{1}{\sqrt{p}}\, dx.$$

where $p = 1 + \alpha x + \beta x^2 + \alpha x^3 + x^4$. We approximate integrals $I = \int_a^b f(x)dx$ with the midpoint rule, namely with the quantity $I_n = h\sum_1^n f(a + \frac{2i-1}{2}h)$, where $h = \frac{b-a}{N}$ and N is the number of subdivisions. We will choose n so that the absolute error

$$|I - I_N| \leq \|f''\|_\infty \frac{(b-a)^3}{24N^2},$$

where $\|\cdot\|_\infty$ is the sup norm in $[a, b]$, is small enough to grant the positivity of the function we study. We denote by Δ_I the absolute error in the estimate of the integral I due to the midpoint rule.

In this particular case, we have the following bounds for the norms of the integrands of $\mathcal{G}_4$ and its first and second derivatives:

$$\left\|\partial_{xx}\tfrac{1}{\sqrt{p}}\right\|_\infty \leq 6+9\alpha+6\alpha^2+5\beta+7\alpha\beta+2\beta^2 \lesssim 2\cdot 10^5$$

$$\left\|\partial_{xx}\partial_\alpha\tfrac{1}{\sqrt{p}}\right\|_\infty \leq 1.5(16+11\alpha+4\alpha^2+7\beta+6\alpha\beta+2\beta^2) \lesssim 2.6\cdot 10^5$$

$$\left\|\partial_{xx}\partial_\beta\tfrac{1}{\sqrt{p}}\right\|_\infty \leq (240+264\alpha+62\alpha^2+72\beta+28\alpha\beta+8\beta^2)/8 \lesssim 10^5$$

$$\left\|\partial_{xx}\partial_{\alpha\alpha}\tfrac{1}{\sqrt{p}}\right\|_\infty \leq \{3(582+333\alpha+24\alpha^2+218\beta+59\alpha\beta+16\beta^2)/4\} \lesssim 1.2\cdot 10^6$$

$$\left\|\partial_{xx}\partial_{\alpha\beta}\tfrac{1}{\sqrt{p}}\right\|_\infty \leq \{3(140+83a+9a^2+35b+10ab+3b^2)/16\} \lesssim 5.2\cdot 10^5$$

$$\left\|\partial_{xx}\partial_{\beta\beta}\tfrac{1}{\sqrt{p}}\right\|_\infty \leq \{3(140+140\alpha+35\alpha^2+30\beta+15\alpha\beta+2\beta^2)/4\} \lesssim 2.2\cdot 10^5$$

$$\left\|\partial_{xx}\partial_{\alpha\alpha\alpha}\tfrac{1}{\sqrt{p}}\right\|_\infty \leq \{15(2400+1128a+96a^2+1008b+288ab+96b^2)/32\} \lesssim 7.8\cdot 10^6$$

$$\left\|\partial_{xx}\partial_{\alpha\alpha\beta}\tfrac{1}{\sqrt{p}}\right\|_\infty \leq \{15(2704+1512a+126a^2+784b+232ab+64b^2)/32\} \lesssim 2.8\cdot 10^6$$

$$\left\|\partial_{xx}\partial_{\alpha\beta\beta}\tfrac{1}{\sqrt{p}}\right\|_\infty \leq \{15(2016+1120a+148a^2+392b+112ab+24b^2)/32\} \lesssim 1.3\cdot 10^6$$

$$\left\|\partial_{xx}\partial_{\beta\beta\beta}\tfrac{1}{\sqrt{p}}\right\|_\infty \leq \{15(1008+1008a+252a^2+168b+84ab+8b^2)/32\} \lesssim 7.1\cdot 10^5$$

In our calculations, in most cases the values of $\mathscr{G}_4$ over R range between 10^{-1} and 10^{-3} so a value of the integration error not larger than 10^{-4} would be desirable. With the norms above, this can be obtained globally in R by taking $N=15$, a value of N still low enough to allow computations to be completed within times of the order of the hour. Note also that the situation is actually even better since the values of these norms in the region where $\mathscr{G}_4$ is closer to zero are actually much lower than the upper bounds above.

Besides the numerical integration errors, of course we have also floating-point arithmetic errors on the evaluation of the integrals $\mathcal{G}_4$ and its derivatives to evaluate the Taylor series truncation and the Taylor seris error. These floating-point errors though are negligible. First, we choose the subdivision of $[0,1]$ so that every extreme x_i of the subdivision is exactly representable in double precision floating precision format. This way we can evaluate exactly the polynomials in the numerator and the denominator $p(x;\alpha,\beta)$, so that the only errors contributing to each summand of the midpoint rule come from the square root of p at the denominator and the division between numerator and denominator.

We implement all our calculations in C/C++ code that we compile with GNU's GCC compiler. Notice that GNU's C Math library does not grant correct rounding for functions besides *sqrt* (square root), *fma* (sums and multiplications) and *rint* (round to nearest integer) [9]. For the evaluation of the root of order n at the denominator of G_n we use therefore GNU's MPFR library, a C library for Multi-Precision Floating-point computations with correct Rounding [7]. This way we can count on the fact that the relative error on the root is the smallest possible, namely 2^{-53}, and, since the numerator is exact, the relative error of the quotient is still 2^{-53}. Going back to the case of $\mathcal{G}_4$, given that the numerators of the integrands are never larger than 8 and that $1 \leq p(x;\alpha,\beta) \leq 1 + 2\alpha + \beta = 308$ in $[0,1] \times R$, the absolute error on a single summand of the Riemann sums is not larger than $8 \cdot 2^{-53} \lesssim 9 \cdot 10^{-16}$ and so the absolute error on the evaluation of an integral due to floating-point errors will not be larger than $2^{15} \cdot 8 \cdot 2^{-53} \lesssim 3 \cdot 10^{-11}$. In short, the error analysis above shows that there is no loss of generality by taking into account only the error coming from the numerical integration method.

Let us illustrate all this in a concrete case, namely the square Q centered at the point $(4.2, 7)$ with side $l = 0.2$. In its most explicit expression, our estimate for the minimum value of $\mathcal{G}_4$ in Q has the form

$$\ell \pm \delta_\ell \pm \Delta_\ell \pm \delta_{\Delta_\ell} \pm \tau \pm \delta_\tau \pm \Delta_\tau \pm \delta_{\Delta_\tau},$$

where ℓ is the minimum value of the linearization of $\mathcal{G}_4$ in Q and τ an upper bound of the corresponding error of the Taylor series truncation of $\mathcal{G}_4$ at the first order – we get τ by evaluating all integrals in the left lower corner of Q and by maximizing the absolute value of the numerator and minimizing the denominator (which is always positive) of $\mathcal{R}_4$ and its derivatives. In other words, $\mathcal{G}_4 > 0$ in Q if

$$\ell - \delta_\ell - \Delta_\ell - \delta_{\Delta_\ell} - \tau - \delta_\tau - \Delta_\tau - \delta_{\Delta_\tau} > 0.$$

In this particular case, $\ell = 1.10... \cdot 10^{-3}$ so all floating point errors can be safely ignored – in fact, the minimum value of ℓ in all cases discussed in the present paper is larger than 10^{-4} so these errors can always be ignored. The error term is $\tau = 9.28... \cdot 10^{-4}$. Within Q, the norms of the integrands in $\mathcal{G}_4$

Set	Size of cells	Fnc.	Sub.	Center	Min. value	Error erms
$\mathbf{n=3}$						
$[2,3.3]^2$	0.01×0.025	H_3	2^{10}	$(2,3.275)$	$2.26\cdot 10^{-3}$	$\tau\lesssim 1.94\cdot 10^{-3}$ $\Delta_{H_3}\lesssim 8\cdot 10^{-5}$ $\Delta_\tau\lesssim 5.2\cdot 10^{-5}$
$[1,5]^2$	0.1×0.1	Γ_3	2^{10}	$(3.1,3.3)$	7.2×10^{-4}	$\tau\lesssim 4.5\times 10^{-4}$ $\Delta_{\Gamma_3}\lesssim 5\cdot 10^{-5}$ $\Delta_\tau\lesssim 10^{-5}$
$[1,21]^2$	0.5×0.5	Γ_3	2^{10}	$(1.25,5.25)$	2.6×10^{-1}	$\tau\lesssim 1.5\cdot 10^{-1}$ $\Delta_\ell\lesssim 3\times 10^{-4}$ $\Delta_\tau\lesssim 3\cdot 10^{-4}$
$\mathbf{n=4}$						
$[3.5,5]\times[5,7]$	0.01×0.025	H_4	2^{15}	$(4.99,5)$	$6.3\cdot 10^{-5}$	$\tau\lesssim 5.6\cdot 10^{-5}$ $\Delta_{H_4}\lesssim 5\cdot 10^{-7}$ $\Delta_\tau\lesssim 1.3\cdot 10^{-6}$
$[2,7]\times[2.4,9]$	0.2×0.2	$\mathcal{G}_4$	2^{13}	$(4.2,7)$	1.1×10^{-3}	$\tau\lesssim 9.2\cdot 10^{-4}$ $\Delta_{\mathcal{G}_4}\lesssim 3\times 10^{-5}$ $\Delta_\tau\lesssim 3\cdot 10^{-5}$
$[2,34]\times[2.4,240]$	0.25×0.5	$\mathcal{G}_4$	2^{11}	$(4.75,8.9)$	8.8×10^{-3}	$\tau\lesssim 1.7\cdot 10^{-3}$ $\Delta_{\mathcal{G}_4}\lesssim 10^{-3}$ $\Delta_\tau\lesssim 5.5\cdot 10^{-3}$
$\mathbf{n=5}$						
$[4,6]\times[9,11]$	0.0125×0.0125	H_4	2^{16}	$(4,1.0975)$	$1.5\cdot 10^{-5}$	$\Delta_{H_4}\lesssim 2.4\cdot 10^{-6}$ $\tau\lesssim 4.9\cdot 10^{-6}$ $\Delta_\tau\lesssim 3.2\cdot 10^{-6}$
$[2.2,11]\times[3,13]$	0.5×0.5	$\mathcal{G}_4$	2^{12}	$(2.2,7)$	4.09×10^{-1}	$\tau\lesssim 2.2\times 10^{-1}$ $\Delta_{\mathcal{G}_5}\lesssim 9\cdot 10^{-3}$ $\Delta_\tau\lesssim 1.2\cdot 10^{-1}$
$[3,50]\times[4,401]$	0.5×1	$\mathcal{G}_4$	2^{13}	$(3,14)$	3.8×10^{-1}	$\tau\lesssim 8.6\times 10^{-2}$ $\Delta_{\mathcal{G}_5}\lesssim 3.4\cdot 10^{-3}$ $\Delta_\tau\lesssim 6.7\cdot 10^{-2}$

Table 1: Essential data on the calculations for the case $n=3$ and the symmetrical cases $n=4,5$. For each n we first verify the positivity of the Hessian of the function (Γ_3, $\mathcal{G}_4$ and $\mathcal{G}_5$ respectively) in a small rectangle (first row) about the known solution ($(3,3)$, $(4,6)$ and $(5,10)$ respectively), then we verify the positivity of the function itself in a larger rectangle (middle row) and then (last row) in the whole region not covered by our analytical arguments (we implicitly assume that, while dealing with a set, we do avoid all points belonging to the sets above). For each case we report the size of the cells used within that set, the function analyzed, the number of subdivisions used to evaluate the integrals through the midpoint rule, the center of the cell where the value of the function was closer to its relative error term, the actual value of the function and, in the rightmost column, the value of the error term coming from the Taylor expansion (τ) and the corresponding errors on the evaluation of the integrals (that we denote by Δ). We do not include the floating-point errors since they are several orders of magnitude smaller.

and its first derivatives are bounded by 10^3 so that, with as little as $N = 10$, we get that $\Delta_\ell, \Delta_\tau \leq 4 \cdot 10^{-5}$. Hence

$$\tau + \Delta_\tau + \Delta_\ell \leq 1.008 \cdot 10^{-3} \leq \ell.$$

When points are too close to the solution $(4, 6)$, this technique cannot work so we rather consider the Hessian

$$H_4 = \partial_{\alpha\alpha}\mathcal{G}_4 \partial_{\beta\beta}\mathcal{G}_4 - (\partial_{\alpha\beta}\mathcal{G}_4)^2$$

and show that it is strictly positive in a convex neighborhood O of it. Within each small cell Q in which we subdivide O we expand H_4 to the zero-th order, namely we use the fact that, for all $(\alpha, \beta) \in Q$, there is a $(\xi, \eta) \in Q$ (in fact, on the segment joining (α_0, β_0) to (α, β)) such that

$$H_4(\alpha, \beta) = H_4(\alpha_0, \beta_0) + \partial_\alpha H_4(\xi, \eta) \cdot (\xi - \alpha_0) + \partial_\beta H_4(\xi, \eta) \cdot (\eta - \beta_0).$$

We get a lower bound for H_4 in Q by maximizing the Taylor series error the same way we did above. Let us denote by H_0, A and B the maxima of, respectively, $H_4(\alpha_0, \beta_0)$, $\partial_\alpha H_4$ and $\partial_\beta H_4$. Similarly to above, our calculations tell us that

$$\min_Q H_4 > H_0 - \delta_{H_0} - \Delta_{H_o} - \delta_{\Delta_{H_o}} - \tau - \delta_\tau - \Delta_\tau - \delta_{\Delta_\tau},$$

where

$$\Delta_{H_0} = \Delta_{\partial_{\alpha\alpha}\mathcal{G}_4} \cdot \max_Q |\partial_{\beta\beta}\mathcal{G}_4| + \max_Q |\partial_{\alpha\alpha}\mathcal{G}_4| \cdot \Delta_{\partial_{\beta\beta}\mathcal{G}_4} +$$
$$+ 2\Delta_{\partial_{\alpha\beta}\mathcal{G}_4} \max_Q |\partial_{\alpha\beta}\mathcal{G}_4|$$

and

$$\Delta_\tau = \Delta_{\partial_{\alpha\alpha\alpha}\mathcal{G}_4} \cdot \max_Q |\partial_{\beta\beta}\mathcal{G}_4| + \max_Q |\partial_{\alpha\alpha\alpha}\mathcal{G}_4| \cdot \Delta_{\partial_{\beta\beta}\mathcal{G}_4} +$$
$$+ \Delta_{\partial_{\alpha\alpha}\mathcal{G}_4} \max_Q |\partial_{\alpha\beta\beta}\mathcal{G}_4| + \max_Q |\partial_{\alpha\alpha}\mathcal{G}_4| \Delta_{\partial_{\alpha\beta\beta}\mathcal{G}_4} + \ldots,$$

so we want to make sure that H_0 is larger than the sum of all error terms. It turns out that, within $O = [3.5, 5] \times [5, 7]$, H_0 is always not smaller than 10^{-5} so that, once again, the floating-point error terms can be disregarded. This time, though, we need a higher precision on the integrals since our bounds for the norms of the second and third derivatives of $\mathcal{G}_4$ are significantly higher than those of $\mathcal{G}_4$ itself and its first derivatives. It is enough to take $N = 15$, so that the error on $\mathcal{G}_4$ and its derivatives up to the third order are bound from above by $3.1 \cdot 10^{-4}$.

Let us consider the concrete case of the rectangle Q centered at the point $(4.99, 5)$. Our calculations show that $H_0 = 6.29... \cdot 10^{-5}$, $\tau = 5.61... \cdot 10^{-5}$, $\Delta_{H_0} = 1.9... \cdot 10^{-6}$ and $\Delta_\tau = 1.3... \cdot 10^{-6}$, so that $\min_Q H_4 > 0$ in Q. Incidentally, this is the cell with the smallest relative minimum value of H_4 with this subdivision.

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